A homogeneous model for P[0] and P[*] nonlinear complementarity problems

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A Homogeneous Model for P_0 and P_* Nonlinear Complementarity Problems

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Abstract

The homogeneous model for linear programs provides a most simple and firm theory in interior point algorithms. In 1999, Andersen and Ye generalized this model to monotone complementarity problems (CPs) and showed that most of desirable properties can be inherited as long as the problem has the monotonicity. However, much dependence on the monotonicity prevents us from extending the model to more general problems, e.g., P_0 CPs or P_* CPs. In this paper, we propose a new homogeneous model and its associated algorithm which have the following features: (a) The homogeneous model preserves the P_0 (P_*) property if the original problem is a P_0 (P_*) CP. (b) The algorithm can be applied to P_0 CPs starting at a positive point near the central trajectory, and it does not need to use any big-$\mathcal{M}$ penalty parameter. (c) The algorithm generates a sequence that approaches feasibility and optimality simultaneously for any P_* CP having a complementarity solution, and (d) it solves the P_* CP having a strictly feasible point.

Key words. P_0 and P_* complementarity problem, homogeneous algorithm, existence of trajectory, global convergence

1 Introduction

This paper deals with the standard complementarity problem (CP)

$$\begin{array}{lll} \text{(CP)} & \text{Find} & (x,s) \in \Re^{2n} \\ & \text{s.t.} & s = f(x), \\ & & (x,s) \geq 0, \\ & & x_i s_i = 0 \ (i \in N), \end{array}$$

where f is a continuously differentiable function from $\Re^n_+ := \{x \in \Re^n : \ x \geq 0\}$ to $\Re^n$ and $N := \{1, 2, \ldots, n\}$.

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A CP is said to be (asymptotically) feasible if and only if there is a bounded sequence $\{(x^k, s^k)\} \in \Re^{2n}_{++}$, $k = 1, 2, \ldots$, such that

$$\lim_{k\to\infty} s^k - f(x^k) = 0$$

where any limit point $(\hat{x}, \hat{s})$ of the sequence is called an (asymptotically) feasible point for the CP. In particular, an (asymptotically) feasible point $(\hat{x}, \hat{s})$ satisfying $(\hat{x}, \hat{s}) > (0, 0)$ is called an interior feasible point or a strictly feasible point.

A CP is said to be (asymptotically) solvable if there is an (asymptotically) feasible point $(\hat{x}, \hat{s})$ such that $\hat{x}^T \hat{s} = 0$ where $(\hat{x}, \hat{s})$ is called a complementarity solution of the CP.

Many interior point algorithms have been developed for the CP where the function f has special properties (see a couple of unprecedentedly comprehensive books by Facchinei and Pang [3, 4], and many other monographs and articles, e.g., [2, 5, 6, 7, 8, 9, 10, 11, 12, 13]). The following is a list of functions which are often used in the literatures:

Definition 1.1 *Let $\mathcal{K}$ be a subset of $\Re^n$ and $\kappa \geq 0$. .*

(i) *A function f is said to be a monotone function from $\mathcal{K}$ to $\Re^n$ if and only if there holds*

$$(x^1 - x^2)^T (f(x^1) - f(x^2)) \geq 0$$

for any $x^1, x^2 \in \mathcal{K}$,

(ii) *A function f is said to be a $P_*(\kappa)$ function from $\mathcal{K}$ to $\Re^n$ if and only if*

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+} (x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) + \sum_{i \in \mathcal{I}_-} (x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) \geq 0$$

for any $x^1, x^2 \in \mathcal{K}$, where $\mathcal{I}_+ := \mathcal{I}_+(x)$ and $\mathcal{I}_- := \mathcal{I}_-(x)$ are a couple of index sets given by

$$\begin{aligned}
\mathcal{I}_+(x) &:= \{i \in N : (x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) \geq 0\}, \\
\mathcal{I}_-(x) &:= \{i \in N : (x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) < 0\}.
\end{aligned}$$

(iii) *A function f is said to be a P_0 function from $\mathcal{K}$ to $\Re^n$ if and only if for any $x^1 \neq x^2 \in \mathcal{K}$, there exists at least one index $i \in N$ such that*

$$(x_i^1 - x_i^2)(f_i(x^1) - f_i(x^2)) \geq 0.$$

We say that the CP is monotone (respectively, $\mathrm{P}_*(\kappa)$ or P_0) if f is monotone (respectively, $\mathrm{P}_*(\kappa)$ or P_0). It can be seen from the definitions above that the monotone CP is a $\mathrm{P}_*(\kappa)$ CP with $\kappa = 0$, and the $\mathrm{P}_*(\kappa)$ CP is a P_0 CP for every $\kappa \geq 0$.

Among other algorithms, Andersen and Ye[1] provide a homogeneous model and an associated algorithm for the monotone CP, having the following features:

(a) The homogeneous model preserves the monotone property if the original problem is a monotone CP.

(b) The algorithm can start at a positive point, feasible or infeasible, near the central trajectory of the positive orthant, and it does not need to use any big-$\mathcal{M}$ penalty parameter.

(c) If the problem is a monotone CP having a complementarity solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously.

(d) The algorithm solves the monotone CP without any regularity assumption concerning the existence of optimal, feasible, or strictly feasible points.

(e) It achieves $\mathcal{O}(\sqrt{n}\log(1/\epsilon))$-iteration complexity if f satisfies a Lipschitz-type smoothness condition.

(f) If the problem is infeasible, the algorithm generates a sequence that converges to a certificate proving infeasibility.

While the algorithm enjoys the properties above, there is a crucial difficulty in applying it to more general problems, since their homogeneous model does not necessarily holds the P_0 (or $\mathrm{P}_*(\kappa)$) property even if the original problem is a P_0 (or $\mathrm{P}_*(\kappa)$) CP. In order to overcome this difficulty, we provide a new homogeneous model and an algorithm for which we can show the following results instead of (a)–(f) above:

(a') The new homogeneous model preserves the P_0 (or $\mathrm{P}_*(\kappa)$) property if the original problem is a P_0 (or $\mathrm{P}_*(\kappa)$) CP.

(b') The algorithm can start at a positive point, feasible or infeasible, near the central trajectory of the positive orthant, and it does not need to use any big-$\mathcal{M}$ penalty parameter.

(c') If the problem is a $\mathrm{P}_*(\kappa)$ CP having a complementarity solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously.

(d') The algorithm solves the $\mathrm{P}_*(\kappa)$ CP if it has a strictly feasible point.

(e') It achieves $O(\sqrt{n}\delta(\kappa,\lambda)\log(1/\epsilon))$-iteration complexity for the $\mathrm{P}_*(\kappa)$ CP if the associated homogeneous function satisfies a Lipschitz-type smoothness condition with a parameter $\lambda \geq 0$. Here $\delta(\kappa,\lambda) \geq \mathcal{O}(1)$ is a parameter which depends on the values κ and λ.

Note that for every $\kappa \geq 0$, the strictly feasibility of the $\mathrm{P}_*(\kappa)$ CP is equivalent to the fact that the (complementarity) solution set of the $\mathrm{P}_*(\kappa)$ CP is nonempty and bounded, which has been shown by Zhao and Li (Theorem 4.2 in [17]).

Our homogeneous model can be regarded as a natural extension of the one proposed by Andersen and Ye[1] in terms of the correspondence between (a)–(c) and (a')–(c') above, while the desirable properties (d)–(f) are restricted to (d') and (e') by abandoning the monotonicity of the problem. To the author's knowledge, this is the first homogeneous model which has the properties (a')–(c'), while related interior point homogeneous algorithms for solving the general CPs have been provided by Lesaja[12, 13].

The paper is organized as follows.

In Section 2, a new homogeneous function ψ and its associated homogeneous model (HCP) are provided. The model preserves the P_0 property if the original problem is a P_0 CP. We also obtain a similar result concerning the $P_*(\kappa)$ property, but it holds true on a specific subset of the domain of the function ψ (Lemma 2.1). The limitation forces us to re-examine some of theoretical tools used in the field of interior point algorithms, which is a second objective of the paper.

In Section 3, we discuss the trajectory induced by the homogeneous model, which plays an essential role in designing interior point algorithms. The existence of the trajectory is ensured if the original problem is a $P_*(\kappa)$ CP having a complementarity solution (Theorem 3.4), while another additional assumption is required to certify that the trajectory leads to a desired solution from which a complementarity solution of the original CP can be calculated (Theorem 3.6). A sufficient condition for the latter part is that the original problem is a strictly feasible $P_*(\kappa)$ CP (Corollary 3.9).

The assumption of the strictly feasibility of the $P_*(\kappa)$ CP also suggests the possibility of designing an algorithm for numerically tracing the trajectory (Theorem 3.8). In Section 4, we give some fundamental results concerning the Jacobian matrix of the homogeneous function in order to provide a Newton-type algorithm. Most of the results in the section are due to the work of Peng, Roos and Terlaky (Chapter 4, [16]).

In Section 5, we describe the details of our homogeneous algorithm. The algorithm consists of a Newton-type direction and an inexact step-size determination, similarly to many other related algorithms ([14, 11, 8], etc.). The algorithm is well-defined whenever the original problem is a P_0 CP and does not need any information about the value κ if the problem is a $P_*(\kappa)$ CP for some $\kappa \geq 0$. The global convergence of the algorithm can be obtained under the assumption that the original problem is a strictly feasible $P_*(\kappa)$ CP (Theorem 5.4).

In Section 6, we discuss the convergence rate of our homogeneous algorithm. To do this, we assume that the homogeneous function ψ satisfies a Lipschitz-type smoothness condition (Assumption 6.1). Under the condition, we can derive an $O(\sqrt{n}\delta_{\kappa,\lambda}\log(1/\epsilon))$-iteration complexity of the algorithm. In general, we may consider that the result shows a local convergence property of the algorithm.

We give some remarks on further research in Section 7.

Here we list some symbols appearing in the paper. We use $\Re^n_+$ and $\Re^n_{++}$ to denote the sets $\{x \in \Re^n : \ x \geq 0\}$ and $\{x \in \Re^n : \ x > 0\}$, respectively. For a given set C, $\mathrm{int}(C)$, $\mathrm{cl}(C)$ and $\partial(C)$ denote the interior of C, the closure of C and the boundary of C, respectively. For the vectors $x \in \Re^n$, $\Delta x \in \Re^n$ and $x^k \in \Re^n$, X, ΔX and X_k denote

$$X := \mathrm{diag}\,\{x_i \ (i \in N)\}, \quad \Delta X := \mathrm{diag}\,\{\Delta x_i \ (i \in N)\}, \quad X_k := \mathrm{diag}\,\{x^k_i \ (i \in N)\}.$$

We often use the following relationships

$$X\Delta x = X(\Delta X)e = (\Delta X)Xe = (\Delta X)x$$

where e denotes the vector whose elements are 1s.

2 A new homogeneous CP model

Andersen and Ye[1] provide the homogeneous *monotone* model (HMCP) related to (CP):

$$\text{(HMCP)} \quad \text{Find} \ \ (x, \tau, s, \kappa) \in \Re^{2(n+1)}$$

$$\text{s.t.} \quad \begin{pmatrix} s \\ \kappa \end{pmatrix} = \begin{pmatrix} \tau f(x/\tau) \\ -x^T f(x/\tau) \end{pmatrix},$$
$$(x, \tau, s, \kappa) \geq 0,$$
$$x_i s_i = 0 \ (i \in N), \tau\kappa = 0.$$

The homogeneous map

$$\begin{pmatrix} \tau f(x/\tau) \\ -x^T f(x/\tau) \end{pmatrix}$$

is monotone on the set $\Re^{n+1}_{++}$ if f is monotone on $\Re^n_+$ (Theorem 1 of [1]). Therefore, many results obtained for the monotone (CP) can be applied to the homogeneous monotone model (HMCP). Unfortunately, this fact does not necessarily hold for general cases. The P_0 (pr $\mathrm{P}_*(\kappa)$) property of the homogeneous model above is not necessarily guaranteed even if f is a P_0 (or a $\mathrm{P}_*(\kappa)$ function. We introduce a new homogeneous model (HCP) below, aiming to extend the results in [1] to the P_0 and the P_* CPs:

$$\text{(HCP)} \quad \text{Find} \ (x, t, s, u) \in \Re^{4n}$$
$$\text{s.t.} \quad \begin{pmatrix} s \\ u \end{pmatrix} = \begin{pmatrix} Tf(T^{-1}x) \\ -Xf(T^{-1}x) \end{pmatrix},$$
$$(x, t, s, u) \geq 0,$$
$$x_i s_i = 0, \ t_i u_i = 0 \ (i \in N).$$

We introduce a new variable $z = (x, t) \in \Re^{2n}$ and define $\psi : \ \Re^n_+ \times \Re^n_{++} \to \Re^{2n}$ as follows:

$$\psi(z) = \psi(x, t) := \begin{pmatrix} Tf(T^{-1}x) \\ -Xf(T^{-1}x) \end{pmatrix}. \tag{1}$$

We use $2N$ to denote the index set $\{1, 2, \ldots, 2n\}$. For every nonempty subset C of $\Re^n_{++}$, we introduce a value τ_C defined by

$$\tau_C := \sup_{t^1, t^2 \in C} \left\{ \frac{\max\{t^1_i t^2_i : \ i \in N\}}{\min\{t^1_i t^2_i : \ i \in N\}} \right\}. \tag{2}$$

Obviously, if the set C is a compact subset of $\Re^n_{++}$ then τ_C has a finite positive value. This can be seen even if the set $C \subset \Re^n_{++}$ is a cone given by

$$C_\Delta := \{\alpha t : \ t \in \Delta, \ \alpha > 0\}$$

where Δ is a compact subset of $\Re^n_{++}$. For an example, let $\pi \in (0, 1)$ and define the set

$$C(\pi) := \{t \in \Re^n_{++} : \ \|t/\alpha - e\|_\infty < \pi, \ \alpha > 0\}. \tag{3}$$

Then we can easily see that

$$C(\pi) \subset C_\delta$$

where Δ is a compact subset of $\Re^n_{++}$ given by

$$\Delta = \{t \in \Re^n_{++} : \ \|t - e\|_\infty < \pi.$$

and in the case $C = C(\pi)$, the value τ_C defined by (2) turns out to be

$$\tau_C = \left(\frac{1+\pi}{1-\pi}\right)^2 > 0.$$

We often use the set $C(\pi)$ with $\pi \in (0,1)$ to derive some important results throughout the paper. The following lemma shows that the new homogeneous function ψ preserves the P_* property of f if we confine the domain of t to a set C for which τ_C has a finite positive value.

Lemma 2.1 **(i)** *If f is a P_0 function from $\Re^n_+$ to $\Re^n$, then ψ is a P_0 function from $\Re^n_+ \times \Re^n_{++}$ to $\Re^{2n}$.*

(ii) *Let C be a subset of $\Re^n_{++}$ having a finite positive value τ_C defined by (2). If f is a $P_*(\kappa)$ function from $\Re^n_+$ to $\Re^n$ for some $\kappa \geq 0$, then ψ is a $P_*(\kappa_C)$ function from $\Re^n_+ \times C$ to $\Re^{2n}$ where κ_C satisfies*

$$1 + 4\kappa_C = \tau_C(1 + 4\kappa). \tag{4}$$

Proof: For each $z^1, z^2 \in \Re^n_+ \times \Re^n_{++}$ and $i \in 2N$, let us define

$$\delta_i := (z^1_i - z^2_i)(\psi_i(z^1) - \psi_i(z^2)).$$

For each $i \in N$, by the definition of ψ, we have,

$$\begin{aligned}
\delta_i &= (x^1_i - x^2_i)(t^1_i f_i(T_1^{-1}x^1) - t^2_i f_i(T_2^{-1}x^2)) \\
&= t^1_i t^2_i (x^1_i - x^2_i)\left(\frac{1}{t^2_i} f_i(T_1^{-1}x^1) - \frac{1}{t^1_i} f_i(T_2^{-1}x^2)\right) \\
&= t^1_i t^2_i \left(\frac{x^1_i}{t^2_i} f_i(T_1^{-1}x^1) - \frac{x^1_i}{t^1_i} f_i(T_2^{-1}x^2) - \frac{x^2_i}{t^2_i} f_i(T_1^{-1}x^1) + \frac{x^2_i}{t^1_i} f_i(T_2^{-1}x^2)\right), \\
\delta_{n+i} &= (t^1_i - t^2_i)(-x^1_i f_i(T_1^{-1}x^1) + x^2_i f_i(T_2^{-1}x^2)) \\
&= t^1_i t^2_i \left(\frac{1}{t^2_i} - \frac{1}{t^1_i}\right)(-x^1_i f_i(T_1^{-1}x^1) + x^2_i f_i(T_2^{-1}x^2)) \\
&= t^1_i t^2_i \left(-\frac{x^1_i}{t^2_i} f_i(T_1^{-1}x^1) + \frac{x^2_i}{t^2_i} f_i(T_2^{-1}x^2) + \frac{x^1_i}{t^1_i} f_i(T_1^{-1}x^1) - \frac{x^2_i}{t^1_i} f_i(T_2^{-1}x^2)\right).
\end{aligned}$$

Thus we can see that for every $i \in N$,

$$\begin{aligned}
(t^1_i t^2_i)^{-1}(\delta_i + \delta_{n+i}) &= -\frac{x^1_i}{t^1_i} f_i(T_2^{-1}x^2) - \frac{x^2_i}{t^2_i} f_i(T_1^{-1}x^1) + \frac{x^2_i}{t^2_i} f_i(T_2^{-1}x^2) + \frac{x^1_i}{t^1_i} f_i(T_1^{-1}x^1) \\
&= \left(\frac{x^1_i}{t^1_i} - \frac{x^2_i}{t^2_i}\right)\left(f_i(T_1^{-1}x^1) - f_i(T_2^{-1}x^2)\right).
\end{aligned} \tag{5}$$

(i): Suppose that f is a P_0 function from $\Re^n_+$ to $\Re^n$. Let $z^1 = (x^1, t^1), z^2 = (x^2, t^2) \in \Re^n_+ \times \Re^n_{++}$ and $(x^1, t^1) \neq (x^2, t^2)$. If $T_1^{-1}x^1 \neq T_2^{-1}x^2$ then, since f is a P_0 function, there exists an index i for which

$$\frac{x_i^1}{t_i^1} - \frac{x_i^2}{t_i^2} \neq 0$$

holds and

$$\left(\frac{x_i^1}{t_i^1} - \frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1) - f_i(T_2^{-1}x^2)\right) \geq 0.$$

By the equality (5), we can see that

$$(t_i^1 t_i^2)^{-1}(\delta_i + \delta_{n+i}) \geq 0$$

and hence

$$\delta_i + \delta_{n+i} \geq 0,$$

which implies that

$$\delta_i \geq 0 \text{ or } \delta_{n+i} \geq 0.$$

On the other hand, if $T_1^{-1}x^1 = T_2^{-1}x^2$ then $\delta_i + \delta_{n+i} = 0$ for every $i \in N$, which leads to the fact

$$\delta_i \geq 0 \text{ or } \delta_{n+i} \geq 0.$$

Therefore, in both cases, there exists an index i such that $\delta_i \geq 0$ or $\delta_{n+i} \geq 0$, i.e., ψ is a P_0 function.

(ii): Suppose that f is a P_* function. For every $z^1 = (x^1, t^1) \in \Re^n_+ \times C$ and every $z^2 = (x^2, t^2) \in \Re^n_+ \times C$, define the index sets

$$\begin{aligned}
\mathcal{I}^1_+ &:= \left\{ i \in N : \delta_i = (z_i^1 - z_i^2)(\psi_i(z^1) - \psi_i(z^2)) \geq 0 \right\}, \\
\mathcal{I}^1_- &:= \left\{ i \in N : \delta_i = (z_i^1 - z_i^2)(\psi_i(z^1) - \psi_i(z^2)) < 0 \right\}, \\
\mathcal{I}^2_+ &:= \left\{ i \in N : \delta_{n+i} = (z_{n+i}^1 - z_{n+i}^2)(\psi_{n+i}(z^1) - \psi_{n+i}(z^2)) \geq 0 \right\}, \\
\mathcal{I}^2_- &:= \left\{ i \in N : \delta_{n+i} = (z_{n+i}^1 - z_{n+i}^2)(\psi_{n+i}(z^1) - \psi_{n+i}(z^2)) < 0 \right\}, \\
\mathcal{I}_+ &:= \left\{ i \in N : \left(\frac{x_i^1}{t_i^1} - \frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1) - f_i(T_2^{-1}x^2)\right) \geq 0 \right\}, \\
\mathcal{I}_- &:= \left\{ i \in N : \left(\frac{x_i^1}{t_i^1} - \frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1) - f_i(T_2^{-1}x^2)\right) < 0 \right\}.
\end{aligned}$$

Then we see that

$$\begin{aligned}
&(1 + 4\kappa_C)\left(\sum_{i \in \mathcal{I}^1_+} \delta_i + \sum_{i \in \mathcal{I}^2_+} \delta_{n+i}\right) + \left(\sum_{i \in \mathcal{I}^1_-} \delta_i + \sum_{i \in \mathcal{I}^2_-} \delta_{n+i}\right) \\
&= (1 + 4\kappa_C)\left(\sum_{i \in \mathcal{I}^1_+} (t_i^1 t_i^2)\frac{\delta_i}{t_i^1 t_i^2} + \sum_{i \in \mathcal{I}^2_+} (t_i^1 t_i^2)\frac{\delta_{n+i}}{t_i^1 t_i^2}\right) + \left(\sum_{i \in \mathcal{I}^1_-} (t_i^1 t_i^2)\frac{\delta_i}{t_i^1 t_i^2} + \sum_{i \in \mathcal{I}^2_-} (t_i^1 t_i^2)\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)
\end{aligned}$$

$$\geq (1+4\kappa_C)\min_{i\in N}\{t_i^1 t_i^2\}\left(\sum_{i\in\mathcal{I}_+^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_+^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)+\max_{i\in N}\{t_i^1 t_i^2\}\left(\sum_{i\in\mathcal{I}_-^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_-^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right).$$

Since we obtain the following inequality by the definitions (4) and (2) of κ_C and τ_C,

$$\begin{aligned}(1+4\kappa_C)\min_{i\in N}\{t_i^1 t_i^2\} &= \tau_C(1+4\kappa)\min_{i\in N}\{t_i^1 t_i^2\}\\ &= \max_{i\in N}\{t_i^1 t_i^2\}\tau_C(1+4\kappa)\frac{\min_{i\in N}\{t_i^1 t_i^2\}}{\max_{i\in N}\{t_i^1 t_i^2\}}\\ &\geq \max_{i\in N}\{t_i^1 t_i^2\}(1+4\kappa),\end{aligned}$$

it follows that

$$\begin{aligned}&(1+4\kappa_C)\left(\sum_{i\in\mathcal{I}_+^1}\delta_i+\sum_{i\in\mathcal{I}_+^2}\delta_{n+i}\right)+\left(\sum_{i\in\mathcal{I}_-^1}\delta_i+\sum_{i\in\mathcal{I}_-^2}\delta_{n+i}\right)\\
&\geq \max_{i\in N}\{t_i^1 t_i^2\}(1+4\kappa)\left(\sum_{i\in\mathcal{I}_+^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_+^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)+\max_{i\in N}\{t_i^1 t_i^2\}\left(\sum_{i\in\mathcal{I}_-^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_-^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)\\
&= \max_{i\in N}\{t_i^1 t_i^2\}\left\{(1+4\kappa)\left(\sum_{i\in\mathcal{I}_+^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_+^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)+\left(\sum_{i\in\mathcal{I}_-^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_-^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)\right\}\\
&= \max_{i\in N}\{t_i^1 t_i^2\}\left\{\left(\sum_{i\in N}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in N}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)+4\kappa\left(\sum_{i\in\mathcal{I}_+^1}\frac{\delta_i}{t_i^1 t_i^2}+\sum_{i\in\mathcal{I}_+^2}\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)\right\}\\
&\geq \max_{i\in N}\{t_i^1 t_i^2\}\left\{\sum_{i\in N}\left(\frac{\delta_i}{t_i^1 t_i^2}+\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)+4\kappa\sum_{i\in\mathcal{I}_+}\left(\frac{\delta_i}{t_i^1 t_i^2}+\frac{\delta_{n+i}}{t_i^1 t_i^2}\right)\right\}\\
&= \max_{i\in N}\{t_i^1 t_i^2\}\left\{\sum_{i\in N}\left(\frac{x_i^1}{t_i^1}-\frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1)-f_i(T_2^{-1}x^2)\right)\right.\\
&\qquad\left.+4\kappa\sum_{i\in\mathcal{I}_+}\left(\frac{x_i^1}{t_i^1}-\frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1)-f_i(T_2^{-1}x^2)\right)\right\}\\
&= \max_{i\in N}\{t_i^1 t_i^2\}\left\{(1+4\kappa)\sum_{i\in\mathcal{I}_+}\left(\frac{x_i^1}{t_i^1}-\frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1)-f_i(T_2^{-1}x^2)\right)\right.\\
&\qquad\left.+\sum_{i\in\mathcal{I}_-}\left(\frac{x_i^1}{t_i^1}-\frac{x_i^2}{t_i^2}\right)\left(f_i(T_1^{-1}x^1)-f_i(T_2^{-1}x^2)\right)\right\}\\
&\geq 0\end{aligned}$$

where the last inequality follows from the facts that f is a P_* function and that $\max_{i\in\mathcal{I}_-}\{t_1^1 t_i^2\}>0$. Thus we conclude that ψ is a $\mathrm{P}_*(\kappa_C)$ function from $\Re^n_+\times C$ to $\Re^{2n}$. ∎

The new homogeneous model (HCP) inherits favorable properties of the homogeneous monotone model in [1]. The following lemma follows from the definition (1) of the new homogeneous function ψ, and does not depend on the P_0 or $P_*(\kappa)$ property of the function f.

Lemma 2.2 **(i)** *For every $z = (x,t) \in \Re^n_+ \times \Re^n_{++}$, $z^T \psi(z) = 0$.*

(ii) *(HCP) is (asymptotically) feasible and every (asymptotically) feasible solution is a complementarity solution.*

(iii) *Let (x^*, t^*, s^*, u^*) be a complementarity solution of (HCP). If $t^* > 0$, then $(T_*^{-1}x^*, T_*^{-1}s^*)$ is a complementarity solution for (CP).*

(iv) *Let $(\hat{x}, \hat{s})$ be a complementarity solution of (CP). Then, for every $t^* > 0$, we can construct an (asymptotically) feasible solution (x^*, t^*, s^*, u^*) i.e., a complementarity solution of (CP) using $(\hat{x}, \hat{s})$.*

Proof: (i): The proof is straightforward.
(ii): Let us take

$$x^k := \left(\frac{1}{2}\right)^k e, \quad t^k := \left(\frac{1}{2}\right)^k e, \quad s^k := \left(\frac{1}{2}\right)^k e, \quad u^k := \left(\frac{1}{2}\right)^k e.$$

Then we can easily see that

$$\lim_{k\to\infty} \left(s^k - T_k f(T_k^{-1}x^k)\right) = \lim_{k\to\infty} \left(\frac{1}{2}\right)^k (e - f(e)) = 0$$

and similarly,

$$\lim_{k\to\infty} \left(u^k + X_k f(T_k^{-1}x^k)\right) = \lim_{k\to\infty} \left(\frac{1}{2}\right)^k (e + f(e)) = 0.$$

Therefore the system is (asymptotically) feasible.

Let $(\hat{x}, \hat{t}, \hat{s}, \hat{u}) \geq 0$ be any (asymptotically) feasible point for (HCP). Then from (i), we see that

$$\hat{x}^T\hat{s} + \hat{t}^T\hat{u} = (\hat{x}^T \hat{t}^T) \begin{pmatrix} \hat{T}f(\hat{T}^{-1}\hat{x}) \\ -\hat{X}f(\hat{T}^{-1}\hat{x}) \end{pmatrix} = 0,$$

which implies that $(\hat{x}, \hat{t}, \hat{s}, \hat{u}) \geq 0$ is a complementarity solution.
(iii): Let (x^*, t^*, s^*, u^*) be a complementarity solution of (HCP) with $t^* > 0$. Then $T_*^{-1}s^* = f(T_*^{-1}x^*)$ and $(s_i^*/t_i^*)(x_i^*/t_i^*) = (x_i^* s_i^*)/(t_i^*)^2 = 0$ $(i \in N)$, i.e., $(T_*^{-1}x^*, T_*^{-1}s^*)$ is a complementarity solution of (CP).
(iv): Let $(\hat{x}, \hat{s})$ be a complementarity solution of (CP) and $t^* > 0$. Define

$$x^k := T_* \left(\hat{x} + \left(\frac{1}{2}\right)^k e\right) > 0, \quad t^k := t^* > 0,$$

$$s^k := T_* \left(\hat{s} + \left(\frac{1}{2}\right)^k e\right) > 0, \quad u^k := \left(\frac{1}{2}\right)^k e > 0.$$

Then, for each $i \in N$, we have

$$\begin{aligned}
\lim_{k\to\infty}\left(s_i^k - t_i^k f_i(T_k^{-1}x^k)\right) &= \lim_{k\to\infty}\left\{t_i^*\left(\hat{s}_i + \left(\frac{1}{2}\right)^k\right) - t_i^* f_i\left(\hat{x} + \left(\frac{1}{2}\right)^k e\right)\right\} \\
&= t_i^*(\hat{s}_i - f_i(\hat{x})) = 0, \\
\lim_{k\to\infty}\left(u_i^k + x_i^k f_i(T_k^{-1}x^k)\right) &= \lim_{k\to\infty}\left\{\left(\frac{1}{2}\right)^k + \left(\hat{x}_i + \left(\frac{1}{2}\right)^k\right) f_i\left(\hat{x} + \left(\frac{1}{2}\right)^k e\right)\right\} \\
&= \hat{x}_i f_i(\hat{x}) = \hat{x}_i \hat{s}_i = 0.
\end{aligned}$$

Thus $(T_*\hat{x}, t^*, T_*\hat{s}, 0)$ is an (asymptotically) feasible solution of (HCP). ▮

Let $z = (x,t) \in \Re^n_+ \times \Re^n_{++}$ and $w = (s,u) \in \Re^{2n}_+$ and define the residual function $r : \Re^n_+ \times \Re^n_{++} \times \Re^{2n}_+ \to \Re^{2n}$ as

$$\begin{aligned}
r_s(z,w) &:= s - Tf(T^{-1}x), \\
r_u(z,w) &:= u + X^{-1}f(T^{-1}x), \\
r(z,w) &:= w - \psi(z) \\
&= (r_s(z,w), r_u(z,w)) \\
&= \left(s - Tf(T^{-1}x), u + Xf(T^{-1}x)\right). \qquad (6)
\end{aligned}$$

As we will see in Section 3, the image $r(\Re^{4n}_{++})$ plays an important role to show the existence of a trajectory for the homogeneous model (HCP). The lemma below gives us a useful fact that the image $r(\Re^{4n}_{++})$ always contain the positive orthant $\Re^{2n}_{++}$ as a subset if the original CP has a complementarity solution.

Lemma 2.3 *Suppose that the original problem (CP) has a complementarity solution. Then, for every nonempty subset C of $\Re^n_{++}$,*

(i) $0 \in cl\left(r(\Re^n_{++} \times C \times \Re^{2n}_{++})\right)$,

(ii) $\tilde{r} + \Re^{2n}_{++} \subset r(\Re^n_{++} \times C \times \Re^n_{++})$ *for every* $\tilde{r} \in r(\Re^n_{++} \times C \times \Re^n_{++})$.

(iii) $\Re^{2n}_+ \subset cl\left(r(\Re^n_{++} \times C \times \Re^{2n}_{++})\right)$,

(iv) $\Re^{2n}_{++} \subset r(\Re^n_{++} \times C \times \Re^{2n}_{++})$.

Proof: (i): Suppose that the original problem (CP) has a complementarity solution. Then the proof of (vi) of Lemma 2.2 implies that there exists a sequence

$$(z^k, w^k) = (x^k, t^*, s^k, u^k) \in \Re^n_{++} \times \{t^*\} \times \Re^{2n}_{++}$$

satisfying

$$\lim_{k\to\infty} r(w^k, z^k) = 0$$

for every $t^* \in \Re^n_{++}$. Thus, we can see that

$$0 \in \mathrm{cl}(r(\Re^n_{++} \times C \times \Re^{2n}_{++}))$$

for every nonempty subset C of $\Re^{2n}_{++}$.
(ii): Let

$$\tilde{r} \in r(\Re^n_{++} \times C \times \Re^n_{++})$$

and let

$$(\tilde{x}, \tilde{t}, \tilde{s}, \tilde{u}) \in \Re^n_{++} \times C \times \Re^n_{++}$$

for which

$$r(\tilde{x}, \tilde{t}, \tilde{s}, \tilde{u}) = \tilde{r}$$

holds. For every $(d_s, d_u) \in \Re^{2n}_{++}$, by the definition (6) of r, we have

$$(\tilde{x}, \tilde{t}, \tilde{s} + d_s, \tilde{u} + d_s) \in \Re^n_{++} \times C \times \Re^n_{++},$$

and

$$r(\tilde{x}, \tilde{t}, \tilde{s} + d_s, \tilde{u} + d_s) = \tilde{r} + (d_s, d_u)$$

which implies the assertion (ii).
(iii): Let

$$(x^k, t^k, s^k, u^k) \in \Re^n_{++} \times C \times \Re^{2n}_{++}$$

be a sequence such that

$$\lim_{k \to \infty} r(x^k, t^k, s^k, u^k) = 0.$$

Then, for every $(d_s, d_u) \in \Re^{2n}_{++}$,

$$(x^k, t^k, s^k + d_s, u^k + d_u) \in \Re^n_{++} \times C \times \Re^{2n}_{++} \quad (k = 1, 2, \ldots)$$

and

$$\lim_{k \to \infty} r(x^k, t^k, s^k + d_s, u^k + d_u) = (d_s, d_u).$$

Thus we have shown (iii).
(iv): Suppose on the contrary that the set $r(\Re^n_{++} \times C \times \Re^n_{++})$ does not contain some $\bar{r} \in \Re^{2n}_{++}$. As we have seen above,

$$\tilde{r} + \Re^{2n}_{++} \subset r(\Re^n_{++} \times C \times \Re^n_{++})$$

for every $\tilde{r} \in r(\Re^n_{++} \times C \times \Re^n_{++})$. Thus

$$\bar{r} \notin r(\Re^n_{++} \times C \times \Re^n_{++})$$

implies that

$$r \notin r(\Re^n_{++} \times C \times \Re^n_{++})$$

for every $r < \bar{r}$ which contradicts the fact

$$0 \in \mathrm{cl}(r(\Re^n_{++} \times C \times \Re^n_{++})).$$

∎

3 Existence of a trajectory to a solution of (HCP)

In this section, we discuss the existence of a trajectory associated with the homogeneous map ψ. First, we observe some key properties of ψ according to the definition (1).

For every $(z, w) = (x, t, s, u) \in \Re^n_+ \times \Re^n_{++} \times \Re^{2n}_+$, the definition (6) of the function r implies the following equation:

$$Xr_s(z, w) + Tr_u(z, w) = X(s - Tf(T^{-1}x)) + T(u + Xf(T^{-1}x)) = Xs + Tu. \tag{7}$$

The next lemma is obtained by a direct calculation from (7), and we often refer to it in the succeeding discussions.

Lemma 3.1 *Let $(z, w) = (x, t, s, u) \in \Re^{4n}_{++}$ satisfying $(b_s, b_u) = (r_s(z, w), r_u(z, w)) \in \Re^{2n}_{++}$. Then, for every $i \in N$, we have*

$$0 < \frac{x_i s_i + t_i u_i - t_i (b_u)_i}{(b_s)_i} = x_i \leq \frac{x_i s_i + t_i u_i}{(b_s)_i},$$
$$0 < \frac{x_i s_i + t_i u_i - x_i (b_s)_i}{(b_u)_i} = t_i \leq \frac{x_i s_i + t_i u_i}{(b_u)_i}$$

for every $i \in N$.

Let us define the function $\Psi : \Re^n_+ \times \Re^n_{++} \times \Re^{2n}_+ \to \Re^{4n}$

$$\begin{aligned}\Psi(z, w) &= (Zw, r(z, w)) \\ &= (Zw, w - \psi(z)) \\ &= \Big(Xs, Tu, s - Tf(T^{-1}x), u + Xf(T^{-1}x)\Big).\end{aligned} \tag{8}$$

For a given vectors $(\bar{a}, \bar{b}) \in \Re^{4n}_{++}$, we consider the following system

$$\Psi(z, w) = \theta(\bar{a}, \bar{b}), \quad (z, w) \in \Re^{4n}_{++} \tag{9}$$

for $\theta \in (0, 1]$.

Lemma 3.1 ensures that if $(z, w) \in \Re^{4n}_{++}$ satisfies (9) for some $\theta \in (0, 1]$, then z lies in a bounded set of $\Re^{2n}_{++}$ which does not depend on the value of $\theta \in (0, 1]$. Note that this does not necessarily imply the existence or the boundedness of the set

$$\{(z, w) = (x, t, s, u) : \ (z, w) \text{ satisfies (9) for some } \theta > 0.\} \tag{10}$$

as it is. However, if the above set forms a bounded trajectory, then we may obtain a bounded sequence $\{(z(\theta^k), w(\theta^k))\} \subset \Re^{4n}_{++}$ with $\theta^k \to 0$, and every accumulation point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of the sequence should be a complementarity solution of the homogeneous model (HCP) by the continuity of Ψ. In addition, if $t^* > 0$ can be obtained, then the solution gives us a complementarity solution of the original problem (CP) as we have shown in Lemma 2.2, which is a desired result.

As we will see below, the set (10) forms a trajectory if f is a P_* function and the original CP has a complementarity solution (see Theorem 3.4). To show the boundedness of the trajectory and to show that $t^* > 0$, we impose a more strict assumption on (HCP), i.e., the existence of a strictly feasible point of the original problem (CP) (see Theorem 3.6 and Theorem 3.8).

Before to proceed, we state several important properties of r, ψ and Ψ for P_0 and P_* functions, some of which are based on the discussions in Kojima, Megiddo and Noma[10] or in Gowda and Tawhid [5]. We use (i), (ii) and (iv) in the lemma below to show Theorem 3.4, while (iii) is required in the proof of Theorem 3.6.

Lemma 3.2 I. *Suppose that the function f is a P_0 function from $\Re^n_+$ to $\Re^n$.*

(i) *Let Ω be a bounded subset of $\Re^{4n}_{++}$ for which there exist constants $\omega_1 > 0$ and $\omega_2 > 0$ satisfying*

$$0 < \omega_1 \le \frac{(a_x)_i + (a_t)_i}{(b_s)_i} \le \omega_2, \quad 0 < \omega_1 \le \frac{(a_x)_i + (a_t)_i}{(b_u)_i} \le \omega_2 \ (i \in N) \tag{11}$$

for every $(a, b) = (a_x, a_t, b_s, b_u) \in \Omega$. If the sequence

$$\{(z^k, w^k) = (x^k, t^k, s^k, u^k) : \ k = 1, 2, \ldots\} \subset \Re^{4n}_{++}$$

satisfies

$$\Psi(z^k, w^k) = (a^k, b^k) = (a^k_x, a^k_t, b^k_s, b^k_u) \in \Omega$$

for every $k = 1, 2, \ldots$, then it is bounded. Moreover, let us define the set of indices

$$\mathcal{I}_t = \{i \in N : \ t^k_i \to 0\}. \tag{12}$$

Then, for every $i \in \mathcal{I}_t$, there exists an infinite subsequence $\{u^k_i\}_{K_i}$ which is bounded.

(ii) *Ψ is one-to-one on $\Re^{4n}_{++}$ which implies that Ψ maps $\Re^{4n}_{++}$ onto $\Psi(\Re^{4n}_{++})$ homeomorphically.*

II. *Let C be an open subset of $\Re^n_{++}$ for which τ_C defined by (2) has a finite positive value. Suppose that the function f is a $P_*(\kappa)$ function from $\Re^n_+$ to $\Re^n$.*

(iii) *Let $(\tilde{z}, \tilde{w}) \in \Re^n_{++} \times C \times \Re^{2n}_{++}$ and let S be a subset of $\Re^n_{++} \times C \times \Re^{2n}_{++}$. Suppose that there exists a constant $\delta > 0$ such that*

$$(z_i - \tilde{z}_i)(\psi_i(z) - \psi_i(\tilde{z})) \le \delta - (\tilde{z}_i w_i + z_i \tilde{w}_i) \tag{13}$$

for every $i \in 2N$ and every $(z, w) \in S$. Then $(z, w) \in S$ satisfies

$$(z, w) \in \Re^{4n}_{++}, \quad \tilde{w}^T z + \tilde{z}^T w \le (1 + 4\kappa_C) 2n\delta$$

which implies that S is bounded. Here κ_C is a nonnegative number defined by (4).

(iv) *For every compact subset D of $\Re^{2n}_+ \times r(\Re^n_{++} \times C \times \Re^{2n}_{++})$, the set $\Psi^{-1}_C(D)$ is bounded. Here $\Psi^{-1}_C(D)$ is given by*

$$\Psi^{-1}_C(D) := \{(z, w) \in \Re^n_{++} \times C \times \Re^{2n}_{++} : \ \Psi(z, w) \in D\}$$

Proof: (i): The boundedness of the sequence $\{z^k\} = \{(x^k, t^k)\}$ follows from Lemma 3.1 and the assumptions. Since $\{z^k\} = \{(x^k, t^k)\}$ is bounded, by taking a subsequence if necessary, we may assume that

$$\begin{aligned}\lim_{k\to+\infty} x_i^k &= \lim \left\{ \frac{(a_x^k)_i + (a_t^k)_i - t_i^k (b_u^k)_i}{(b_s^k)_i} \right\} \\ &\geq \omega_1 > 0\end{aligned}$$

and that $x_i^k/t_i^k \to +\infty$ for every $i \in \mathcal{I}_t$. Let us construct a bounded sequence $\{v^k\} \subset \Re^n_+$ as

$$v_i^k = \begin{cases} 0 & (i \in \mathcal{I}_t), \\ \bar{x}_i^k/\bar{t}_i^k & (i \notin \mathcal{I}_t) \end{cases}$$

for every k. Since f is a P_0 function from $\Re^n_+$ to $\Re^n$ and since $v^k \neq T_k^{-1} x^k$ $(k = 1, 2, \ldots)$, there exist an index $j \in \mathcal{I}_t$ and an infinite subset $K_j \subset \{1, 2, \ldots\}$ for which

$$(x_j^k/t_j^k)\left(f_j(T_k^{-1}x^k) - f_j(v^k)\right) = (x_j^k/t_j^k - v_j^k)\left(f_j(T_k^{-1}x^k) - f_j(v^k)\right) \geq 0$$

holds for every $k \in K_j$. The above inequality implies that $\{f_j(T_k^{-1}x^k)\}_{K_j}$ is bounded below by $\inf_{k\in K} f_j(v^k) > -\infty$. Since $\{(b_u^k)_j\}$ and $\{x_j^k\}$ are bounded, the relation

$$0 < u_j^k = (b_u^k)_j - x_j^k f_j(T_k^{-1}x^k)$$

implies that $\{u_j^k\}_{K_j}$ is bounded.
(ii): Assume that $\Psi(z^1, w^1) = \Psi(z^2, w^2)$ for some distinct $(z^1, w^1) \in \Re^{4n}_{++}$ and $(z^2, w^2) \in \Re^{4n}_{++}$. Then, by the definition (8) of Ψ, we have

$$\psi(z^1) - \psi(z^2) = w^1 - w^2,\ z_i^1 w_i^1 = z_i^2 w_i^2 > 0 \ \ (i \in 2N).$$

Since ψ is a P_0 function by (i) of Lemma 2.1, there exists an index i such that

$$z_i^1 \neq z_i^2, \ \ 0 \leq (z_i^1 - z_i^2)(\psi_i(z^1) - \psi_i(z^2)) = (z_i^1 - z_i^2)(w_i^1 - w_i^2)$$

We may assume that $z_i^1 > z_i^2$. Then the above inequality implies that $w_i^1 \geq w_i^2$, which contradicts the assumption $z_i^1 w_i^1 = z_i^2 w_i^2 > 0$.

The homeomorphism follows from the domain invariance theorem, i.e., a continuous one-to-one mapping Ψ from $\Re^{4n}_{++}$ into $\Re^{4n}$ maps open sets into open sets.
(iii): Since the value τ_C is positive and finite, from Lemma 2.1, ψ is a $\mathrm{P}_*(\kappa_C)$ function on the set $\Re^n_+ \times C$, i.e., for every $z \in \Re^n_{++}$ and $\tilde{z} \in \Re^n_{++} \times C$, we have

$$0 \leq (1 + 4\kappa_C) \sum_{i\in\mathcal{I}_+} (z_i - \tilde{z}_i)(\psi_i(z) - \psi_i(\tilde{z})) + \sum_{i\in\mathcal{I}_-} (z_i - \tilde{z}_i)(\psi_i(z) - \psi_i(\tilde{z})) \tag{14}$$

where

$$\begin{aligned}\mathcal{I}_+ &= \{i \in 2N : \ (z_i - \tilde{z}_i)(\psi_i(z) - \psi_i(\tilde{z})) \geq 0\}, \\ \mathcal{I}_- &= \{i \in 2N : \ (z_i - \tilde{z}_i)(\psi_i(z) - \psi_i(\tilde{z})) < 0\}.\end{aligned}$$

Substituting (13) into (14), we see that

$$\begin{aligned}
0 &\le (1+4\kappa_C)\sum_{i\in\mathcal{I}_+}(z_i-\tilde z_i)(\psi_i(z)-\psi_i(\tilde z))+\sum_{i\in\mathcal{I}_-}(z_i-\tilde z_i)(\psi_i(z)-\psi_i(\tilde z))\\
&\le (1+4\kappa_C)\sum_{i\in\mathcal{I}_+}[\delta-(\tilde z_i w_i+z_i\tilde w_i)]+\sum_{i\in\mathcal{I}_-}[\delta-(\tilde z_i w_i+z_i\tilde w_i)]\\
&\le (1+4\kappa_C)2n\delta-(1+4\kappa_C)\sum_{i\in\mathcal{I}_+}(\tilde z_i w_i+z_i\tilde w_i)-\sum_{i\in\mathcal{I}_-}(\tilde z_i w_i+z_i\tilde w_i)\\
&\le (1+4\kappa_C)2n\delta-\sum_{i\in\mathcal{I}_+}(\tilde z_i w_i+z_i\tilde w_i)-\sum_{i\in\mathcal{I}_-}(\tilde z_i w_i+z_i\tilde w_i)\\
&\le (1+4\kappa_C)2n\delta-(\tilde z^T w+z^T\tilde w)
\end{aligned}$$

and hence

$$\tilde z^T w+z^T\tilde w\le(1+4\kappa_C)2n\delta.$$

Since $(z,w)\in\Re^{4n}_{++}$, the above inequality implies that

$$S\subset\{(z,w)\in\Re^{4n}_{++}:\ \tilde w^T z+\tilde z^T w\le(1+4\kappa_C)2n\delta\}.$$

By the positivity of $(\tilde z,\tilde w)$, the set $\{(z,w)\in\Re^{4n}_{++}:\ \tilde w^T z+\tilde z^T w\le(1+4\kappa_C)2n\delta\}$ is bounded. Thus we obtain the assertion.

(iv): Suppose that $\Psi_C^{-1}(D)$ is unbounded. We may take a sequence

$$\{(z^k,w^k)\in\Psi_C^{-1}(D):k=1,2,\dots\}$$

such that

$$\lim_{k\to\infty}\|(z^k,w^k)\|=\infty.$$

Let $(a^k,b^k):=(Z_k w^k,w^k-\psi(z^k))=\Psi(z^k,w^k)\in D$ for every $k=1,2,\dots$. By the compactness of the set $D\subset\Re^{2n}_{+}\times r(\Re^n_{++}\times C\times\Re^{2n}_{++})$, there exists a $\bar b\in r(\Re^n_{++}\times C\times\Re^{2n}_{++})$ for which we have

$$\lim_{k\to\infty}b^k=\bar b\in r(\Re^n_{++}\times C\times\Re^{2n}_{++}).$$

Since C is an open subset of $\Re^n_{++}$, as we have shown in (ii) above, the set

$$\Psi(\Re^n_{++}\times C\times\Re^{2n}_{++})$$

is open in $\Re^{4n}$ and hence the set

$$r(\Re^n_{++}\times C\times\Re^{2n}_{++})$$

is also open in $\Re^{2n}$. Thus, there exists a $\tilde b$ such that

$$\tilde b\in r(\Re^n_{++}\times C\times\Re^{2n}_{++}) \tag{15}$$

and

$$b^k=w^k-\psi(z^k)\ge\tilde b$$

for every sufficiently large k. Note that (15) implies the existence of $(\tilde z,\tilde w)$ satisfying

$$(\tilde z,\tilde w)\in\Re^n_{++}\times C\times\Re^{2n}_{++}\ \text{ and }\ \tilde w-\psi(\tilde z)=\tilde b.$$

Since $w^k - b^k = \psi(z^k)$ and $\tilde{w} - \tilde{b} = \psi(\tilde{z})$, by a simple calculation, we have

$$\begin{aligned}
&(z_i^k - \tilde{z}_i)(\psi_i(z^k) - \psi_i(\tilde{z})) \\
&= (z_i^k - \tilde{z}_i)((w_i^k - b_i^k) - (\tilde{w}_i - \tilde{b}_i)) \\
&= z_i^k w_i^k - \tilde{z}_i w_i^k - z_i^k(b_i^k + \tilde{w}_i - \tilde{b}_i) + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i) \\
&= a_i^k - \tilde{z}_i w_i^k - z_i^k(b_i^k + \tilde{w}_i - \tilde{b}_i) + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i)
\end{aligned} \tag{16}$$

for each $i \in 2N$. Using the facts $z^k \geq 0$ and $b^k \geq \tilde{b}$, the above inequality can be deduced by

$$\begin{aligned}
&a_i^k - \tilde{z}_i w_i^k - z_i^k(b_i^k + \tilde{w}_i - \tilde{b}_i) + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i) \\
&\leq a_i^k - \tilde{z}_i w_i^k - z_i^k \tilde{w}_i + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i) \\
&= a_i^k + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i) - \tilde{z}_i w_i^k - z_i^k \tilde{w}_i.
\end{aligned} \tag{17}$$

By the boundedness of D, there exists a $\delta > 0$ for which

$$a_i^k + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i) \leq \delta$$

holds for every k and $i \in 2N$, and hence we have

$$a_i^k + \tilde{z}_i(b_i^k + \tilde{w}_i - \tilde{b}_i) - \tilde{z}_i w_i^k - z_i^k \tilde{w}_i \leq \delta - (\tilde{z}_i w_i^k + z_i^k \tilde{w}_i). \tag{18}$$

Combining the above inequalities (16), (17) and (18), we can conclude that

$$(z_i^k - \tilde{z}_i)(\psi_i(z^k) - \psi_i(\tilde{z})) \leq \delta - (\tilde{z}_i w_i^k + z_i^k \tilde{w}_i) \tag{19}$$

for every k and $i \in 2N$. By (iii) above, the inequality (19) implies that the boundedness of the set $\{(z^k, w^k)\}$, which contradicts the assumption $\|(z^k, w^k)\| \to \infty$. ∎

To show the existence of a trajectory, we impose the following assumption on (CP):

Assumption 3.3 **(i)** *The original problem (CP) has a complementarity solution* $(\hat{x}, \hat{s})$.

(ii) f *is a* $P_*(\kappa)$ *function from* $\Re^n_+$ *to* $\Re^n$.

The theorem below guarantees the existence of such trajectory under Assumption 3.3. Note that the existence of a strictly feasible solution of the original problem is not required to show the theorem.

Theorem 3.4 *Suppose that Condition 3.3 holds.*

(i) *For every* $(a, b) \in \Re^{4n}_{++}$,

$$\Psi(z, w) = (a, b)$$

has a unique solution $(z, w) \in \Re^{4n}_{++}$.

(ii) $\Re^{4n}_{++} \subset \Psi(\Re^{4n}_{++})$.

(iii) *Let* $(\bar{a}, \bar{b}) \in \Re^{4n}_{++}$ *and define the target line (segment)* $\mathcal{T}$ *as*

$$\mathcal{T} := \{\theta(\bar{a}, \bar{b}) : \ \theta > 0\}. \tag{20}$$

For every $\theta \in (0, 1]$, *the system (9) has a unique solution* $(z(\theta), w(\theta))$ *which is continuous in* θ. *Thus*

$$\Psi^{-1}(\mathcal{T}) := \{(z, w) \in \Re^{4n}_{++} : \ \Psi(z, w) = \theta(\bar{a}, \bar{b}), \ \theta \in (0, 1]\}$$

forms a trajectory.

Proof:
(i): Let $\{\pi^k \in (0, 1) : \ k = 1, 2, \ldots\}$ be a strictly decreasing sequence satisfying

$$\lim_{k\to\infty} \pi^k = 0.$$

Define the set $C^k = C(\pi^k)$ for each $k = 1, 2, \ldots$, where $C(\pi^k)$ is defined by (3). For every k, the set C^k is a nonempty open subset of $\Re^n_{++}$ and τ_{C^k} defined by (2) with $C = C^k$ has a finite positive value, i.e.,

$$\tau_{C^k} = \left(\frac{1 + \pi^k}{1 - \pi^k}\right)^2 > 0.$$

Since $\pi^k \in (0, 1)$ is strictly decreasing, for every k and k' with $k < k'$, we see that

$$\tau_{C^k} < \tau_{C^{k'}}$$

and hence

$$(\mathrm{cl}C^k \setminus \{0\}) \subset C^{k'} \subset \Re^n_{++}. \tag{21}$$

Let $(a, b) \in \Re^{2n}_{+} \times \Re^{2n}_{++}$ and set $k = 1$. By (iii) of Lemma 2.3, we know that

$$b \in \Re^{2n}_{++} \subset r(\Re^n_{++} \times C^1 \times \Re^{2n}_{++})$$

which implies the existence of a $(\hat{z}, \hat{w})$ satisfying

$$(\hat{z}, \hat{w}) \in \Re^n_{++} \times C^1 \times \Re^{2n}_{++} \ \text{ and } \ b = \hat{w} - \psi(\hat{z}).$$

Let us define

$$\hat{a} := \hat{Z}\hat{w} > 0$$

and

$$D_a := \{(1 - \theta)\hat{a} + \theta a : \ \theta \in [0, 1]\}. \tag{22}$$

Consider a family of systems

$$\Psi(z, w) = ((1 - \theta)\hat{a} + \theta a, b), \ \ (z, w) \in \Re^n_{++} \times C^1 \times \Re^{2n}_{++} \tag{23}$$

for $\theta \in [0, 1]$. Since $D_a \subset \Re^{2n}_{+}$ and $D := D_a \times \{b\}$ is a compact subset of $\Re^{2n}_{+} \times r(\Re^{4n}_{++})$, the set $\Psi^{-1}_{C^1}(D)$ is bounded by (iv) of Lemma 3.2. Define

$$\bar{\theta}^1 := \sup\{\tilde{\theta} \in [0, 1] : \ (23) \text{ has a solution for any } \theta \in [0, \tilde{\theta}] \ \}. \tag{24}$$

Since the set $\Psi(\Re^n_{++} \times C^1 \times \Re^{2n}_{++})$ is open by (ii) of Lemma 3.2, $(a,b) \in \Psi(\Re^n_{++} \times C^1 \times \Re^{2n}_{++})$ implies that $\bar{\theta}^1 > 0$. By the definition of $\bar{\theta}^1$, there exists a sequence $\{(z^p, w^p, \theta^p)\}$ which satisfies

$$\Psi(z^p, w^p) = ((1-\theta^p)\hat{a} + \theta^p a, b), \quad (z^p, w^p) \in \Re^n_{++} \times C^1 \times \Re^{2n}_{++}$$

and $\theta^p \to \bar{\theta}^1$. Note that the sequence $\{(z^p, w^p)\}$ is contained the bounded set $\Psi^{-1}_{C^1}(D)$. Thus, we may assume that there exist a $(\bar{z}^1, \bar{w}^1)$ to which the sequence $\{(z^p, w^p)\}$ converges. By the continuity of the function Ψ, the limit point $(\bar{z}^1, \bar{w}^1, \bar{\theta}^1)$ also satisfies the system

$$\Psi(\bar{z}^1, \bar{w}^1) = ((1-\bar{\theta}^1)\hat{a} + \bar{\theta}^1 a, b), \quad (\bar{z}^1, \bar{w}^1) \in \mathrm{cl}(\Re^n_{++} \times C^1 \times \Re^{2n}_{++}).$$

Since

$$\bar{z}^1_i \bar{w}^1_i = (1-\bar{\theta}^1)\hat{a}_i + \bar{\theta}^1 a_i > 0 \ (i \in 2N)$$

whenever $\bar{\theta}^1 \in [0,1]$, we see that

$$(\bar{z}^1, \bar{w}^1) \in \Re^{4n}_{++}.$$

Therefore, if $\bar{\theta}^1 = 1$ then we obtain a desired result.

Suppose that $\bar{\theta}^1 < 1$. Then the fact

$$\begin{aligned}(\bar{z}^1, \bar{w}^1) &\in \Re^{4n}_{++} \cap \mathrm{cl}(\Re^n_{++} \times C^1 \times \Re^{2n}_{++}) \\ &\subset \Re^n_{++} \times (\mathrm{cl}C^1 \setminus \{0\}) \times \Re^{2n}_{++}.\end{aligned}$$

which we have seen above implies that

$$(\bar{z}^1, \bar{w}^1) \in \Re^n_{++} \times (\partial C^1 \setminus \{0\}) \times \Re^{2n}_{++} \tag{25}$$

where ∂C^1 denotes the boundary of the set C^1. In fact, if not then $(\bar{z}^1, \bar{w}^1)$ lies in the open subset $\Re^n_{++} \times C^1 \times \Re^{2n}_{++}$ of $\Re^{4n}_{++}$, and by the local homeomorphism of Ψ on $\Re^{4n}_{++}$ (see (ii) of Lemma 3.2), the system (23) has a solution for every θ sufficiently close to $\bar{\theta}^1$, which contradicts the definition (24) of $\bar{\theta}^1$.

Next, let us consider the nonempty open set C^2 instead of C^1. By the property (21) of the sets $C^k (k = 1, 2, \ldots)$, the relation (25) implies that

$$(\bar{z}^1, \bar{w}^1) \in \Re^n_{++} \times C^2 \times \Re^{2n}_{++}.$$

Let us consider the new system

$$\Psi(z, w) = ((1-\theta)\hat{a} + \theta a, b), \ (z, w) \in \Re^n_{++} \times C^2 \times \Re^{2n}_{++} \tag{26}$$

and define

$$\bar{\theta}^2 := \sup\{\tilde{\theta} \in [0,1] : \ (26) \text{ has a solution for any } \theta \in [0, \tilde{\theta}] \ \}.$$

Since $(\bar{z}^1, \bar{w}^1)$ lies in the open subset $\Re^n_{++} \times C^2 \times \Re^{2n}_{++}$ of $\Re^{4n}_{++}$, we obtain the inequalities

$$0 < \bar{\theta}^1 < \bar{\theta}^2 \leq 1.$$

By a similar discussion above, we can conclude that a desired result $\bar{\theta}^2 = 1$ or a point $(\bar{z}^2, \bar{w}^2, \bar{\theta}^2)$ satisfying

$$\begin{aligned}&\bar{\theta}^2 < 1, \\ &\Psi(\bar{z}^2, \bar{w}^2) = ((1-\bar{\theta}^2)\hat{a} + \bar{\theta}^2 a, b), \\ &(\bar{z}^2, \bar{w}^2) \in \Re^n_{++} \times (\partial C^2 \setminus \{0\}) \times \Re^{2n}_{++}\end{aligned}$$

is obtained.

By repeating this process, we finally obtain

Case 1: $\bar{\theta}^k = 1$ for some finite $k \in \{1, 2, \ldots\}$

or

Case 2: an infinite sequence $(\bar{z}^k, \bar{w}^k, \bar{\theta}^k)$ satisfying

$$\begin{aligned}
&\bar{\theta}^k < 1, \\
&\Psi(\bar{z}^k, \bar{w}^k) = ((1-\bar{\theta}^k)\hat{a} + \bar{\theta}^k a, b) > 0 && (27)\\
&(\bar{z}^k, \bar{w}^k) = (\bar{x}^k, \bar{t}^k, \bar{s}^k, \bar{u}^k) \in \Re^n_{++} \times (\partial C^k \setminus \{0\}) \times \Re^{2n}_{++} && (28)
\end{aligned}$$

for every $k = 1, 2, \ldots$.

The former case guarantees that the assertion holds. In what follows, we assume that the latter case occurs.

Let us denote $a = (a_x, a_t) \in \Re^{2n}_{++}$, $\hat{a} = (\hat{a}_x, \hat{a}_t) \in \Re^{2n}_{++}$ and $b = (b_x, b_t) \in \Re^{2n}_{++}$. It follows from the relations (7) and (27) that

$$\bar{X}_k b_s + \bar{T}_k b_u = \bar{X}_k \bar{s}^k + \bar{T}_k \bar{u}^k = (1-\bar{\theta}^k)(\hat{a}_x + \hat{a}_t) + \bar{\theta}^k (a_x + a_t)$$

for every $k = 1, 2, \ldots$. Since $\{\bar{\theta}^k\}$ is a bounded subset of $(0, 1)$, the positivity of (b_s, b_u) ensures the boundedness of the sequence $\{(\bar{x}^k, \bar{t}^k)\}$. Combining this with the fact

$$\bar{t}^k \in (\partial C^k \setminus \{0\}) \quad (k = 1, 2, \ldots)$$

in (28), we may assume that there exists an index $i \in N$ for which $\bar{t}^k_i \to 0$. Note that the equations

$$\begin{aligned}
\bar{x}^k_i \bar{s}^k_i &= (1-\bar{\theta}^k)(\hat{a}_x)_i + \bar{\theta}^k (a_x)_i \quad (i \in N), \\
\bar{t}^k_i \bar{u}^k_i &= (1-\bar{\theta}^k)(\hat{a}_t)_i + \bar{\theta}^k (a_t)_i \quad (i \in N)
\end{aligned}$$

and the fact $\bar{\theta}^k \in (0, 1)$ imply that

$$\begin{aligned}
\bar{x}^k_i \bar{s}^k_i &\geq \min\{(\hat{a}_x)_i, (a_x)_i, (\hat{a}_t)_i, (a_t)_i\} > 0, \\
\bar{t}^k_i \bar{u}^k_i &\geq \min\{(\hat{a}_x)_i, (a_x)_i, (\hat{a}_t)_i, (a_t)_i\} > 0
\end{aligned}$$

for every $k = 1, 2, \ldots$ and they do not go to 0 for every $i \in N$. Thus the relation

$$\bar{t}^k_i \to 0$$

implies

$$\bar{u}^k_i \to +\infty.$$

Since the function f is a P_0 function by (ii) of Assumption 3.3, the above deduction leads to a contradiction to the result (i) of Lemma 3.2. Thus Case 2 never occurs and we obtain the desired result.

The uniqueness of the solution follows from the homeomorphism of the function Ψ as we have shown in (ii) of Lemma 3.2.
(ii): It is straightforward from the assertion (ii) .
(iii): The assertion follows from (ii) of Lemma 3.2 and (iii) above. ∎

The above theorem ensures the existence of a trajectory $\Psi^{-1}(\mathcal{T})$. To show the boundedness of the trajectory, we should impose an additional assumption on the trajectory $\Psi^{-1}(\mathcal{T})$.

Assumption 3.5 *There exists an open subset C for which τ_C defined by (2) has a finite positive value and*

$$\Psi^{-1}(\mathcal{T}) \subset \Re^n_{++} \times C \times \Re^{2n}_{++}.$$

In the next theorem, we show that the trajectory $\Psi^{-1}(\mathcal{T})$ is bounded and every limit point (x^*, t^*, s^*, u^*) of the trajectory is a complementarity solution of (HCP) with $t^* > 0$ if Assumptions 3.3 and 3.5 hold.

Theorem 3.6 *Suppose that Assumptions 3.3 and 3.5 are satisfied.*

(i) *The trajectory $\Psi^{-1}(\mathcal{T})$ is bounded.*

(ii) *Every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\Psi^{-1}(\mathcal{T})$ is a complementarity solution of (HCP) with $t^* > 0$.*

Proof: Let $\theta \in (0, 1]$ be fixed and let $(z, w) = (x, t, s, u)$ be the unique solution $(z(\theta), w(\theta)) \in \Psi^{-1}(\mathcal{T})$ of (9). Note that (z, w) satisfies

$$\Psi(z, w) = (Zw, w - \psi(z)) = (\theta\bar{a}, \theta\bar{b})$$

and

$$0 < Z\bar{b} < (e^T\bar{a})e \tag{29}$$

by the relation (7).
(i): Let $(z^1, w^1) = (z(1), w(1)) \in \Re^{4n}_{++}$. Substituting $z_i w_i = \theta\bar{a}_i$, $w_i - \psi_i(z) = \theta\bar{b}_i$, $z_i^1 w_i^1 = \bar{a}_i$, $w_i^1 - \psi_i(z^1) = \bar{b}_i$, we have

$$\begin{aligned}
&(z_i - z_i^1)(\psi_i(z) - \psi_i(z^1)) \\
&= (z_i - z_i^1)[(w_i - \theta\bar{b}_i) - (w_i^1 - \bar{b}_i)] \\
&= (z_i - z_i^1)[(w_i - w_i^1) + (1 - \theta)\bar{b}_i] \\
&= (z_i - z_i^1)(w_i - w_i^1) + (1 - \theta)(z_i - z_i^1)\bar{b}_i \\
&= [z_i w_i - (z_i w_i^1 + z_i^1 w_i) + z_i^1 w_i^1] + (1 - \theta)z_i\bar{b}_i - (1 - \theta)z_i^1\bar{b}_i \\
&= [\theta\bar{a}_i - (z_i w_i^1 + z_i^1 w_i) + \bar{a}_i] + (1 - \theta)z_i\bar{b}_i - (1 - \theta)z_i^1\bar{b}_i \\
&= (1 + \theta)\bar{a}_i + (1 - \theta)z_i\bar{b}_i - (1 - \theta)z_i^1\bar{b}_i - (z_i w_i^1 + z_i^1 w_i) \\
&\le (1 + \theta)\bar{a}_i + (1 - \theta)z_i\bar{b}_i - (z_i w_i^1 + z_i^1 w_i)
\end{aligned}$$

where the last inequality follows from $\theta \in (0, 1]$, $z^1 > 0$ and $b > 0$. Thus, by (29), we can find a constant $\delta = 3(e^T a) > 0$ which satisfies

$$(z_i - z_i^1)(\psi_i(z) - \psi_i(z^1)) \le \delta - (z_i w_i^1 + z_i^1 w_i)$$

for every $i \in N$. Since Assumption 3.5 implies that $(z, w) \in \Re^n_{++} \times C \times \Re^{2n}_{++}$, by (iii) of Lemma 3.2, (z, w) lies in a bounded set which does not depend on the value of $\theta \in (0, 1]$.

(ii): Let $(\bar{z}^*, \bar{w}^*) = (\bar{x}^*, \bar{t}^*, \bar{s}^*, \bar{u}^*)$ be a complementarity solution of (HCP) with $\bar{t}^* > 0$, whose existence has been ensured by (vi) of Lemma 2.2. By a similar calculation above, we have

$$\begin{aligned}
&(z_i - \bar{z}_i^*)(\psi_i(z) - \psi_i(\bar{z}^*)) \\
&= (z_i - \bar{z}_i^*)[(w_i - \theta \bar{b}_i) - \bar{w}_i^*] \\
&= (z_i - \bar{z}_i^*)[(w_i - \bar{w}_i^*) - \theta \bar{b}_i] \\
&= (z_i - \bar{z}_i^*)(w_i - \bar{w}_i^*) - \theta(z_i - \bar{z}_i^*)\bar{b}_i \\
&= [z_i w_i - (z_i \bar{w}_i^* + \bar{z}_i^* w_i) + \bar{z}_i^* \bar{w}_i^*] - \theta z_i \bar{b}_i + \theta \bar{z}_i^* \bar{b}_i \\
&= [\theta \bar{a}_i - (z_i \bar{w}_i^* + \bar{z}_i^* w_i)] - \theta z_i \bar{b}_i + \theta \bar{z}_i^* \bar{b}_i \\
&\leq \theta(\bar{a}_i + \bar{z}_i^* \bar{b}_i) - (z_i \bar{w}_i^* + \bar{z}_i^* w_i).
\end{aligned} \tag{30}$$

Let us define

$$\delta = \theta \max\{\bar{a}_i + \bar{z}_i^* \bar{b}_i : \ i \in 2N\} > 0. \tag{31}$$

By (30) and (iii) of Lemma 3.2 again, we obtain that

$$\sum_{i \in 2N} (\bar{z}_i^* w_i + z_i \bar{w}_i^*) \leq (1 + 4\kappa_C) 2n\delta.$$

Since $Zw = \theta \bar{a}$, substituting $z_i = \theta \bar{a}_i / w_i$ and $w_i = \theta \bar{a}_i / z_i$, we have

$$\sum_{i \in 2N} \theta \bar{a}_i \left(\frac{\bar{z}_i^*}{z_i} + \frac{\bar{w}_i^*}{w_i} \right) \leq (1 + 4\kappa_C) 2n\delta.$$

By the definition (31) of δ, the above inequality implies that

$$\sum_{i \in 2N} \left(\frac{\bar{z}_i^*}{z_i} + \frac{\bar{w}_i^*}{w_i} \right) \leq (1 + 4\kappa_C) 2n \frac{\max\{\bar{a}_i + \bar{z}_i^* \bar{b}_i : \ i \in 2N\}}{\min\{\bar{a}_i : \ i \in 2N\}}.$$

Denoting the right hand side of the inequality as $\bar{\delta} > 0$, we can see that

$$\frac{\bar{z}_i^*}{z_i} \leq \bar{\delta}, \quad \frac{\bar{w}_i^*}{w_i} \leq \bar{\delta} \quad (i \in N),$$

or equivalently

$$\frac{\bar{x}_i^*}{x_i} \leq \bar{\delta}, \quad \frac{\bar{t}_i^*}{t_i} \leq \bar{\delta}, \quad \frac{\bar{s}_i^*}{s_i} \leq \bar{\delta}, \quad \frac{\bar{u}_i^*}{u_i} \leq \bar{\delta} \quad (i \in N)$$

for every $(z, w) = (x, t, s, u) \in \Psi^{-1}(\mathcal{T})$. The above inequalities ensure that any limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\Psi^{-1}(\mathcal{T})$ satisfies

$$(x^*, t^*, s^*, u^*) \geq (\bar{x}^* \bar{\delta}, \bar{t}^* \bar{\delta}, \bar{s}^* \bar{\delta}, \bar{u}^* \bar{\delta})$$

and we can conclude that $t^* > 0$ since $\bar{t}^* > 0$ and $\bar{\delta} > 0$. ∎

Now our interest is how to find whether Assumption 3.5 is satisfied or not. The following theorem gives us not only an answer to this question but an idea for constructing an algorithm to numerically trace the trajectory, which we describe in Section 5. Here we introduce another assumption.

Assumption 3.7 **(i)** *The original problem (CP) has a strictly feasible point* $(\bar{x}, \bar{s})$.

(ii) f *is a* $P_*(\kappa)$ *function from* $\Re^n_+$ *to* $\Re^n$.

It should be noted that Assumption 3.7 implies Assumption 3.3 (see, e.g., [10]). In addition, the theorem below shows that Assumption 3.7 also implies Assumption 3.5 as a corollary (Corollary 3.9).

Theorem 3.8 *Suppose that Assumption 3.7 holds. Let* Ω *be a bounded subset of* $\Re^{4n}_{++}$ *for which there exist constants* $\omega_1 > 0$ *and* $\omega_2 > 0$ *satisfying the inequalities in (11) and*

$$0 < \omega_1 \leq \frac{(a_t)_i}{(b_u)_i} \leq \omega_2 \quad (i \in N) \tag{32}$$

for every $(a, b) = (a_x, a_t, b_s, b_u) \in \Omega$. *Then there exists a set* $C \subset \Re^n_{++}$ *for which* τ_C *defined by (2) has a finite positive value and*

$$\Psi^{-1}(\Omega) \subset \Re^n_{++} \times C \times \Re^{2n}_{++}. \tag{33}$$

More precisely, the set C *above is given by* $C = C(\pi)$ *for some* $\pi \in (0, 1)$. *Here the set* $C(\pi)$ *is defined by (3).*

Proof: Since Assumption 3.7 implies Assumption 3.3 as we have described above, Theorem 3.4 holds under the new assumption. By (i) and (ii) of the theorem, we see that

$$\Omega \subset \Re^{4n}_{++} \subset \Psi(\Re^{4n}_{++}) \text{ and } \Psi^{-1}(\Omega) \subset \Re^{4n}_{++}.$$

Suppose on the contrary that there is no $C \subset \Re^n_{++}$ with a finite positive τ_C for which the relation (33) holds. Let $\{\pi^k \in (0, 1) : \ k = 1, 2, \ldots\}$ be a strictly decreasing sequence such that

$$\lim_{k \to \infty} \pi^k = 0.$$

Define the set $C^k = C(\pi^k)$ where $C(\pi)$ is defined by (3). Since

$$\Psi^{-1}(\Omega) \not\subset \Re^n_{++} \times C^k \times \Re^{2n}_{++}$$

for every k, we obtain a sequence $\{(z^k, w^k) = (x^k, t^k, s^k, u^k)\} \subset \Re^{4n}_{++}$ such that

$$\Psi(z^k, w^k) \in \Omega$$

but

$$t^k \notin C^k$$

for every $k = 1, 2, \ldots$. Here the assumption on the set Ω and (i) of Lemma 3.2 ensure the boundedness of $\{z^k = (x^k, t^k)\}$. Thus, by a similar discussion as in the proof of (i) of Theorem 3.4, we may assume that there exists an index $i \in N$ for which $t^k_i \to 0$, and by taking a subsequence if necessary,

$$\lim_{k \to \infty} z^k = \tilde{z} = (\tilde{x}, \tilde{t}) \in \Re^{2n}_{+}.$$

Let us define the set of indices

$$\mathcal{I}_t = \{i : \ t_i^k \to 0\}.$$

It follows from the observation above that

$$\mathcal{I}_t \neq \emptyset$$

and

$$\lim_{k\to\infty} t_j^k = \tilde{z}_j > 0 \ \ (j \notin \mathcal{I}_t). \tag{34}$$

Denote

$$(a^k, b^k) = (a_x^k, a_t^k, b_s^k, b_u^k) = \Psi(z^k, w^k) = (X_k s^k, T_k u^k, r_s(z^k, w^k), r_u(z^k, w^k))$$

for every $k = 1, 2, \ldots$. By (7), we see that

$$x_i^k (b_s^k)_i + t_i^k (b_u^k)_i = (a_x^k)_i + (a_t^k)_i > 0 \ \ (i \in N).$$

Since $(a^k, b^k) = (a_x^k, a_t^k, b_s^k, b_u^k) \in \Omega$ for every k, by the inequality (11) (and by taking a subsequence again), we have

$$\lim_{k\to\infty} x_i^k \geq \omega_1 > 0 \ \ (i \in \mathcal{I}_t) \tag{35}$$

and hence

$$\lim_{k\to\infty} x_i^k / t_i^k = +\infty \ \ (i \in \mathcal{I}_t). \tag{36}$$

Let $j \in \mathcal{I}_t$. By (i) of Lemma 3.2, there exists an infinite subsequence $\{u_j^k\}_{K_j}$ which is bounded. Note that for every other $i \in \mathcal{I}_t$, we still have

$$t_i^k \underset{K}{\to} 0.$$

Thus, by taking subsequences finitely many times, we may find a subsequence $\{z^k = (x^k, t^k)\}_{\bar{K}}$ for which $\{u_i^k\}_{\bar{K}}$ is bounded for every $i \in \mathcal{I}_t$. Along the subsequence $\{(x^k, t^k)\}_{\bar{K}}$, we have

$$t_i^k u_i^k = (a_t^k)_i \underset{\bar{K}}{\to} 0 \ \ (i \in \mathcal{I}_t).$$

Therefore, by the inequality (32), we must have

$$u_i^k + x_i^k f_i(T_k^{-1} x^k) = (b_u^k)_i \underset{\bar{K}}{\to} 0 \ \ (i \in \mathcal{I}_t).$$

Since $\{u_i^k\}_{\bar{K}} \subset \Re_{++}^n$ is bounded, this implies the boundedness of $\{x^k f_i(T_k^{-1} x^k)\}_{\bar{K}}$ and any subsequence $\{x^k f_i(T_k^{-1} x^k)\}_{\hat{K}}$ of $\{x^k f_i(T_k^{-1} x^k)\}_{\bar{K}}$ should satisfy

$$\lim_{k \underset{\hat{K}}{\to} \infty} x^k f_i(T_k^{-1} x^k) \leq 0 \ \ (i \in \mathcal{I}_t).$$

Similarly, since $\{x_i^k\}_{\bar{K}} \subset \Re_{++}^n$ is bounded and satisfies (35), we also see the boundedness of $\{f_i(T_k^{-1} x^k)\}_{\hat{K}}$ and the following relation

$$\lim_{k \underset{\hat{K}}{\to} \infty} f_i(T_k^{-1} x^k) \leq 0 \ \ (i \in \mathcal{I}_t). \tag{37}$$

Let $(\bar{x}, \bar{s}) \in \Re^{2n}_{++}$ be the strictly feasible point of (CP) whose existence is ensured by the assumption. Since $\bar{s} > 0$ and $f_i(\bar{x}) = \bar{s}_i$, by (36) and (37), we have

$$\begin{aligned}\lim_{\substack{k \to \infty \\ \hat{K}}} (x^k_i/t^k_i - \bar{x}_i)(f_i(T_k^{-1}x^k) - f_i(\bar{x})) &= \lim_{\substack{k \to \infty \\ \hat{K}}} (x^k_i/t^k_i - \bar{x}_i)(f_i(T_k^{-1}x^k) - \bar{s}_i) \\ &= -\infty \quad (i \in \mathcal{I}_t). \qquad (38)\end{aligned}$$

On the other hand, since (34) holds for every $j \notin \mathcal{I}_t$, we see that the sequences $\{x^k_j/t^k_j\}_{\hat{K}}$ and $\{u^k_j = (a^k_t)_j/t^k_j\}_{\hat{K}}$ are bounded, and by the equation

$$x^k_j f_j(T_k^{-1}x^k) = (b^k_u)_j - u^k_j,$$

the sequences $\{x^k_j f_j(T_k^{-1}x^k)\}_{\hat{K}}$ and $\{(x^k_j/t^k_j) f_j(T_k^{-1}x^k)\}_{\hat{K}}$ are also bounded. Moreover, combining the facts $s^k \in \Re^n_{++}$ and (34)with the equation

$$f_j(T_k^{-1}x^k) = [s^k_j - (b^k_s)_j]/t^k_j,$$

$\{f_j(T_k^{-1}x^k)\}_{\hat{K}}$ should be bounded below. Thus, by the equation

$$(x^k_j/t^k_j - \bar{x}_j)(f_j(T_k^{-1}x^k) - \bar{s}_j) = (x^k_j/t^k_j) f_j(T_k^{-1}x^k) - \bar{x}_j f_j(T_k^{-1}x^k) - (x^k_j/t^k_j - \bar{x}_j)\bar{s}_j,$$

we can conclude that there exists a constant δ for which

$$(x^k_j/t^k_j - \bar{x}_j)(f_j(T_k^{-1}x^k) - \bar{s}_j) \le \delta \quad (j \notin \mathcal{I}_t,\ k \in \hat{K}). \qquad (39)$$

From observations (38) and (39), it can be deduced that along the sequence $\{T_k^{-1}x^k\}_{\hat{K}} \subset \Re^n_{++}$, there is no constant $\kappa > 0$ for which

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}_+} (x^k_i/t^k_i - \bar{x}_i)(f_i(T_k^{-1}x^k) - f_i(\bar{x})) + \sum_{i \in \mathcal{I}_-} (x^k_i/t^k_i - \bar{x}_i)(f_i(T_k^{-1}x^k) - f_i(\bar{x})) \ge 0$$

holds where

$$\begin{aligned}\mathcal{I}_+ &:= \{i \in N : (x^k_i/t^k_i - \bar{x}_i)(f_i(T_k^{-1}x^k) - f_i(\bar{x})) \ge 0\}, \\ \mathcal{I}_- &:= \{i \in N : (x^k_i/t^k_i - \bar{x}_i)(f_i(T_k^{-1}x^k) - f_i(\bar{x})) < 0\}.\end{aligned}$$

This is a contradiction to the P_* property of the function f on the set $\Re^n_+$. ∎

It is easy to see that the following corollary follows from Theorems 3.6 and 3.8, since the set $\mathcal{T}$ defined by (20) satisfies (11) and (32) with

$$\begin{aligned}\omega_1 &= \min\left\{\frac{(\bar{a}_x)_i + (\bar{a}_t)_i}{(\bar{b}_s)_i},\ \frac{(\bar{a}_x)_i + (\bar{a}_t)_i}{(\bar{b}_u)_i},\ \frac{(\bar{a}_t)_i}{(\bar{b}_u)_i} \quad (i \in N)\right\}, \\ \omega_2 &= \max\left\{\frac{(\bar{a}_x)_i + (\bar{a}_t)_i}{(\bar{b}_s)_i},\ \frac{(\bar{a}_x)_i + (\bar{a}_t)_i}{(\bar{b}_u)_i},\ \frac{(\bar{a}_t)_i}{(\bar{b}_u)_i} \quad (i \in N)\right\}\end{aligned}$$

for the given $(\bar{a}, \bar{b}) \in \Re^{4n}_{++}$.

Corollary 3.9 *Suppose that Assumption 3.7 holds. Then*

(i) *the trajectory $\Psi^{-1}(\mathcal{T})$ is bounded, and*

(ii) *every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\Psi^{-1}(\mathcal{T})$ is a complementarity solution of (HCP) with $t^* > 0$.*

4 The Jacobian matrix of the function Ψ

Theorem 3.8 implies that if we choose a suitable neighborhood Ω of the target segment $\mathcal{T}$ and generate a sequence $\{(z^k, w^k) \in \Re^{4n} : \ k = 1, 2, \ldots\}$ such that

$$\Psi(z^k, w^k) \in \Omega \ \text{ and } \ \Psi(z^k, w^k) \to 0,$$

then there exists a subset $C \in \Re^n_{++}$ for which

$$\{(z^k, w^k)\} \subset \Re^n_{++} \times C \times \Re^{2n}_{++}$$

and we may find a desirable solution whenever Assumption 3.7 holds. This fact gives us a motivation to construct an algorithm for numerically tracing the trajectory $\Psi^{-1}(\mathcal{T})$. In this section, to provide an algorithm based on the Newton's method, we propose some results concerning the Jacobian matrix of the function ψ.

By its definition (1), the function ψ is continuously differentiable on the set $\Re^{4n}_{++}$ by the assumption that f is continuously differentiable. The Jacobian matrix $\nabla\psi(z)$ of ψ at $z \in \Re^{2n}_{++}$ is given by

$$\nabla\psi(z) = \begin{pmatrix} T\nabla f(y)T^{-1} & F(y) - T\nabla f(y)T^{-2}X \\ -F(y) - X\nabla f(y)T^{-1} & X\nabla f(y)T^{-2}X \end{pmatrix}, \tag{40}$$

where

$$y := T^{-1}x \ \text{ and } \ F(y) := \operatorname{diag}\{f_i(y) \ (i \in N)\}. \tag{41}$$

Corollary 4.7 below, which is the main result of this section, can be obtained by a direct calculation as in the proof of Lemma 2.1. Instead of repeating the calculation, we show more general results concerning the Jacobian matrices of P_0 and P_* functions. In what follows, we define the sets of indices $\mathcal{I}^M_+(x)$ and $\mathcal{I}^M_-(x)$ as

$$\begin{aligned} \mathcal{I}^M_+(x) &:= \{i \in N : x_i[Mx]_i \geq 0\}, \\ \mathcal{I}^M_-(x) &:= \{i \in N : x_i[Mx]_i < 0\}, \end{aligned}$$

for an $n \times n$ matrix M and $x \in \Re^n$, and also define $\mathcal{I}^\phi_+(x^1, x^2)$ and $\mathcal{I}^\phi_-(x^1, x^2)$ as

$$\begin{aligned} \mathcal{I}^\phi_+(x^1, x^2) &:= \{i \in N : (x^1_i - x^2_i)(\phi_i(x^1) - \phi_i(x^2)) \geq 0\}, \\ \mathcal{I}^\phi_-(x^1, x^2) &:= \{i \in N : (x^1_i - x^2_i)(\phi_i(x^1) - \phi_i(x^2)) < 0\} \end{aligned}$$

for a given function $\phi : \Re^n \to \Re^n$ and $x^1, x^2 \in \Re^n$. We first collect some definitions which appear in this section as follows.

Definition 4.1 *Let $\mathcal{K}$ be a subset of $\Re^n$, $\kappa \geq 0$ and $\beta \geq 0$.*

(i) *A matrix $M \in \Re^{n\times n}$ is said to be a P_0 matrix if and only if for any $x \neq 0 \in \Re^n$, there exists at least one index $i \in N$ such that $x_i(Mx)_i \geq 0$.*

(ii) *A matrix $M \in \Re^{n\times n}$ is said to be a $P_*(\kappa)$ matrix if and only if there holds*

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}^M_+(x)} x_i[Mx]_i + \sum_{i \in \mathcal{I}^M_-(x)} x_i[Mx]_i \geq 0$$

for any $x \in \Re^n$.

(iii) *A matrix $M \in \Re^{n\times n}$ is said to be a $P_*(\kappa,\beta)$ matrix if and only if there holds*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+^M(x)} x_i[Mx]_i + \sum_{i\in\mathcal{I}_-^M(x)} x_i[Mx]_i \geq \beta\|x\|^2$$

for any $x \in \Re^n$.

(iv) *A function ϕ is said to be a strict $P_*(\kappa)$ function from $\mathcal{K}$ to $\Re^n$ if and only if there holds*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+^\phi(x^1,x^2)} (x_i^1-x_i^2)\left(\phi_i(x^1)-\phi_i(x^2)\right) + \sum_{i\in\mathcal{I}_-^\phi(x^1,x^2)} (x_i^1-x_i^2)\left(\phi_i(x^1)-\phi_i(x^2)\right) > 0$$

for any $x^1 \neq x^2 \in \mathcal{K}$.

(v) *A function ϕ is said to be a $P_*(\kappa,\beta)$ function from $\mathcal{K}$ to $\Re^n$ if and only if there holds*

$$(1+4\kappa)\sum_{i\in\mathcal{I}_+^\phi} (x_i^1-x_i^2)\left(\phi_i(x^1)-\phi_i(x^2)\right) + \sum_{i\in\mathcal{I}_-^\phi} (x_i^1-x_i^2)\left(\phi_i(x^1)-\phi_i(x^2)\right) \geq \beta\,\|x-y\|^2$$

for any $x^1 \neq x^2 \in \mathcal{K}$.

The proposition below has been essentially given by Moré and Rheinboldt [15] and shown as a more general result in the comprehensive survey by Facchinei and Pang [3].

Proposition 4.2 (3.5.9 Proposition of [3]) *A continuously differentiable function $\phi : \Re_{++}^n \to \Re^n$ is a P_0 function if and only if $\nabla\phi(x)$ is a P_0 matrix for each $x \in \Re_{++}^n$.*

An extension of the above result to the case where ψ is a P_* function from $\Re^n$ to $\Re^n$ has been done by Peng, Roos and Terlaky [16]. The following lemmas are essentially the same as the ones in [16], except for that the domain of the function ϕ is restricted to an open and convex subset $\mathcal{K}$ of $\Re^n$. We omit their proofs, which can be derived by following the ones of the results indicated in the lemmas. See also Lesaja [12, 13] for another results concerning P_* function.

Lemma 4.3 (Lemma 4.2.5 of [16]) *Let $\mathcal{K}$ be a subset of $\Re^n$ and κ be a nonnegative constant. Then a function $f\phi(x) : \mathcal{K} \to \Re^n$ is a $P_*(\kappa)$ function if and only if for any positive $\beta > 0$, the function $\phi_\beta(x) = \phi(x) + \beta x$ is a $P_*(\kappa,\beta)$ function.*

Lemma 4.4 (Lemma 4.2.8 of [16]) *Let $\mathcal{K}$ be an open convex subset of $\Re^n$, $\kappa \geq 0$ and $\beta > 0$. Suppose that $\phi(x) : \mathcal{K} \to \Re^n$ is continuously differentiable. If $\phi(x)$ is a $P_*(\kappa)$ (or $P_*(\kappa,\beta)$) function, then for any $x \in \mathcal{K}$, $\nabla\phi(x)$ is a $P_*(\kappa)$ (or $P_*(\kappa,\beta)$) matrix.*

Lemma 4.5 (Lemma 4.2.9 of [16]) *Let $\mathcal{K}$ be an open convex subset of $\Re^n$, $\kappa \geq 0$ and $\beta > 0$. Suppose that $\phi(x) : \mathcal{K} \to \Re^n$ is continuously differentiable. Suppose that for any $x \in \mathcal{K}$, the Jacobian matrix $\nabla\phi(x)$ is a $P_*(\kappa,\beta)$ matrix. Then ϕ is a strict $P_*(\kappa)$ function on $\mathcal{K}$.*

Proposition 4.6 (Proposition 4.2.10 of [16]) *Let $\mathcal{K}$ be an open convex subset of $\Re^n$ and $\kappa \geq 0$. Suppose that $\phi(x) : \mathcal{K} \to \Re^n$ is continuously differentiable. Then ϕ is a $P_*(\kappa)$ function on $\|$ if and only if for any $x \in \mathcal{K}$, the Jacobian matrix $\nabla\phi(x)$ is a $P_*(\kappa)$ matrix.*

As a corollary of Proposition 4.6, a desired result follows from (ii) of Lemma 2.1.

Corollary 4.7 *Suppose that f is a P_* function from $\Re_+^n$ to $\Re^n$. Let $C \subset \Re_{++}^n$ be an open convex set for which τ_C defined by (2) has a finite positive value. Then the Jacobian $\mathcal{D}\psi(z)$ of the function ψ defined by (1) is a $P_*(\kappa_C)$ matrix for any $z \in \Re_{++}^n \times C$.*

5 A globally convergent homogeneous algorithm

In this section, we provide a homogeneous algorithm for numerically tracing the trajectory $\Psi^{-1}(\mathcal{T})$.

Let $(z^0, w^0) = (x^0, t^0, s^0, u^0) \in \Re^{4n}_{++}$ be a point satisfying

$$r^0 := w^0 - \psi(z^0) > 0$$

and

$$\|Z^0 w^0 - \mu^0 e\| < \beta \mu^0$$

where

$$\mu^0 := \frac{(z^0)^T w^0}{2n}$$

and $\beta \in (0,1)$ is a given constant. For any $\beta \in (0,1)$, such a point $(z^0, w^0) = (x^0, t^0, s^0, u^0)$ can be easily obtained, e.g.,

$$z^0 = e, \quad w^0 = 2 \max_{i \in N}\{|\psi_i(z^0)|\} e.$$

At each iteration k with $(z^k, w^k) := (x^k, t^k, s^k, u^k)$, we set

$$r^k := w^k - \psi(z^k), \quad \text{and} \quad \mu^k := \frac{(z^k)^T w^k}{2n} \tag{42}$$

and calculate a direction $(\Delta z^k, \Delta w^k)$ by solving the following system of linear equations:

$$\Delta w^k - \nabla \psi(z^k) \Delta z^k = -\eta r^k, \tag{43}$$

$$Z_k \Delta w^k + W_k \Delta z^k = \gamma \mu^k e - Z_k w^k. \tag{44}$$

Here $\eta \in (0,1)$ and $\gamma \in (0,1)$ are given constants. The Jacobian matrix $\nabla \psi(z^k)$ is given by (40) with $z = z^k$, which satisfies

$$(z^k)^T \nabla \psi(z^k) = -\psi(z^k)^T \quad \text{and} \quad (z^k)^T \nabla \psi(z^k) z^k = -\psi(z^k)^T z^k = 0 \tag{45}$$

where the latter inequality follows from (i) of Lemma 2.2. Using the equalities above, we obtain the following lemma.

Lemma 5.1 *Suppose that f is a P_0 function.*

(i) *The system of (43) and (44) has a unique solution for every $(z^k, w^k) \in \Re^{4n}_{++}$.*

(ii) *The direction $(\Delta z^k, \Delta w^k)$ satisfies*

$$(\Delta z^k)^T \Delta w^k = (\Delta z^k)^T \nabla \psi(z^k) \Delta z^k + \eta(1 - \eta - \gamma) 2n\mu$$

Proof: (i): The Jacobian matrix $\nabla \psi(z^k)$ is a P_0 matrix by Proposition 4.2. This implies that the coefficient matrix

$$\begin{pmatrix} I & -\nabla \psi(z^k) \\ Z_k & W_k \end{pmatrix} \tag{46}$$

of the system is nonsingular for any $(z^k, w^k) \in \Re^{4n}_{++}$ (see, e.g., Lemma 4.1 of [9]).
(ii): The proof is analogous to the one of Lemma 2 in [1]. Premultiplying each side of (43) by $(\Delta z^k)^T$ gives

$$(\Delta z^k)^T \Delta w^k - (\Delta z^k)^T \nabla\psi(z^k)\Delta z^k = -\eta(\Delta z^k)^T(w^k - \psi(z^k)) \tag{47}$$

and doing so by $(z^k)^T$ gives

$$\begin{aligned}(z^k)^T \Delta w^k + \psi(z^k)^T \Delta z^k &= -\eta(z^k)^T(w^k - \psi(z^k)) \\ &= -\eta(z^k)^T w^k \\ &\qquad\qquad\qquad\qquad \text{(by (45))} \\ &= -\eta 2n\mu^k. \\ &\qquad\qquad\qquad\qquad \text{(by (42))}\end{aligned} \tag{48}$$

On the other hand, premultiplying each side of (44) by e^T gives

$$\begin{aligned}(z^k)^T \Delta w^k + (w^k)^T \Delta z^k &= \gamma\mu^k e^T e - (z^k)^T w^k \\ &= \gamma\mu^k 2n - 2n\mu^k \\ &= -(1-\gamma)2n\mu^k.\end{aligned} \tag{49}$$

Thus we have

$$\begin{aligned}(\Delta z^k)^T \Delta w^k &= (\Delta z^k)^T \nabla\psi(z^k)\Delta z^k - \eta(\Delta z^k)^T(w^k - \psi(z^k)) \\ &\qquad\qquad\qquad \text{(by (47))} \\ &= (\Delta z^k)^T \nabla\psi(z^k)\Delta z^k - \eta((\Delta z^k)^T w^k - (\Delta z^k)^T \psi(z^k)) \\ &= z(\Delta z^k)^T \nabla\psi(z^k)\Delta z^k - \eta\{(-(1-\gamma)2n\mu^k - (z^k)^T \Delta w^k) - (-\eta 2n\mu^k - (z^k)^T \Delta w^k)\} \\ &\qquad\qquad\qquad \text{(by (49) and (48))} \\ &= (\Delta z^k)^T \nabla\psi(z^k)\Delta z^k + \eta(1-\eta-\gamma)2n\mu^k.\end{aligned}$$

∎

Analogously to Andersen and Ye's homogeneous algorithm, we find a new iterate $(z^k(\alpha), w^k(\alpha))$ so that the residual is definitely decreased with a step-size $\alpha \in (0,1)$. Let us define

$$z^k(\alpha) := z^k + \alpha\Delta z^k > 0, \tag{50}$$
$$w^k(\alpha) := w^k + \alpha\Delta w^k + g^k(\alpha), \tag{51}$$
$$r^k(\alpha) := w^k(\alpha) - \psi(z^k(\alpha)), \tag{52}$$
$$\mu^k(\alpha) := \frac{(z^k(\alpha))^T w^k(\alpha)}{2n}, \tag{53}$$

where

$$g^k(\alpha) := \psi(z^k(\alpha)) - \psi(z^k) - \alpha\nabla\psi(z^k)\Delta z^k. \tag{54}$$

The function $w^k(\alpha)$ above is a generalization of the one proposed in [14] for the convex programming problem, and also used in [11, 8], etc. By similar calculations as in [1], we obtain the following lemma.

Lemma 5.2 *Let $\eta + \gamma = 1$. Suppose that (z^k, w^k) satisfies*

$$\|Z_k w^k - \mu^k e\| \leq \beta \mu^k \tag{55}$$

Then the new iterate $(z^k(\alpha), w^k(\alpha))$ given by (50) and (51) satisfies

(i) $r^k(\alpha) = (1 - \alpha\eta) r^k$,

(ii) $\mu^k(\alpha) = (1 - \alpha\eta)\mu^k$,

(iii)

$$\|Z_k(\alpha) w^k(\alpha) - \mu^k(\alpha) e\| \leq \beta \mu^k(\alpha) \left\{ (1 - \alpha\gamma) + \alpha^2 \frac{\|\Delta Z_k \Delta w^k\|}{\beta(1 - \alpha\eta)\mu^k} + \frac{\|Z_k(\alpha) g^k(\alpha)\|}{\beta(1 - \alpha\eta)\mu^k} \right\}$$

(iv)

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \alpha^2 \frac{\|\Delta Z_k \Delta w^k\|}{\beta(1 - \alpha\eta)\mu^k} + \frac{\|Z_k(\alpha) g^k(\alpha)\|}{\beta(1 - \alpha\eta)\mu^k} \right\} = 0$$

Proof: (i): The assertion is obtained by a direct calculation:

$$\begin{aligned}
r^k(\alpha) &= w^k(\alpha) - \psi(z^k(\alpha)) \\
&\qquad\qquad \text{(by (52))} \\
&= w^k + \alpha \Delta w^k + g^k(\alpha) - \psi(z^k(\alpha)) \\
&\qquad\qquad \text{(by (51))} \\
&= w^k + \alpha \Delta w^k - \psi(z^k) - \alpha \nabla\psi(z^k)\Delta z^k \\
&\qquad\qquad \text{(by (54))} \\
&= (w^k - \psi(z^k)) + \alpha(\Delta w^k - \nabla\psi(z^k)\Delta z^k) \\
&= r^k + \alpha(-\eta r^k) \\
&\qquad\qquad \text{(by (42) and (43))} \\
&= (1 - \alpha\eta) r^k.
\end{aligned}$$

(ii): Since the term $(z^k(\alpha))^T g^k(\alpha)$ turns out to be

$$\begin{aligned}
(z^k(\alpha))^T g^k(\alpha) &= (z^k(\alpha))^T (\psi(z^k(\alpha)) - \psi(z^k) - \alpha \nabla\psi(z^k)\Delta z^k) \\
&\qquad\qquad \text{(by (54))} \\
&= (z^k(\alpha))^T (-\psi(z^k) - \alpha \nabla\psi(z^k)\Delta z^k) \\
&\qquad\qquad \text{(by (i) of Lemma 2.2))} \\
&= (z^k + \alpha \Delta z^k)^T (-\psi(z^k) - \alpha \nabla\psi(z^k)\Delta z^k) \\
&= -\alpha^2 (\Delta z^k)^T \nabla\psi(z^k)\Delta z^k \qquad (56) \\
&\qquad\qquad \text{(by (i) of Lemma 2.2 and (45))}
\end{aligned}$$

we have

$$
\begin{aligned}
& 2n\mu^k(\alpha) \\
& = (z^k(\alpha))^T w^k(\alpha) \\
& = (z^k(\alpha))^T (w^k + \alpha \Delta w^k + g^k(\alpha)) \\
& = (z^k(\alpha))^T (w^k + \alpha \Delta w^k) - \alpha^2 (\Delta z^k)^T \nabla \psi(z^k) \Delta z^k \\
& \qquad \qquad \qquad \qquad \text{(by (56))} \\
& = (z^k + \alpha \Delta z^k)^T (w^k + \alpha \Delta w^k) - \alpha^2 (\Delta z^k)^T \nabla \psi(z^k) \Delta z^k \\
& = (z^k)^T w^k + \alpha ((z^k)^T \Delta w^k + (w^k)^T \Delta z^k) + \alpha^2 ((\Delta z^k)^T \Delta w^k - (\Delta z^k)^T \nabla \psi(z^k) \Delta z^k) \\
& = (z^k)^T w^k + \alpha e^T (\gamma \mu^k e - Z_k w^k) + \alpha^2 \eta (1 - \eta - \gamma) 2n\mu^k \\
& \qquad \qquad \qquad \qquad \text{(by (43) and (ii) of Lemma 5.1)} \\
& = (z^k)^T w^k + \alpha e^T (\gamma \mu^k e - Z_k w^k) \\
& \qquad \qquad \qquad \qquad \text{(by } 1 - \eta - \gamma = 0) \\
& = \{1 - \alpha(1 - \gamma)\} 2n\mu^k \\
& \qquad \qquad \qquad \qquad \text{(by (42))} \\
& = (1 - \alpha\eta) 2n\mu^k. \\
& \qquad \qquad \qquad \qquad \text{(by } 1 - \eta - \gamma = 0)
\end{aligned}
$$

(iii): Since the vector $Z_k(\alpha) w^k(\alpha) - \mu^k(\alpha) e$ turns out to be

$$
\begin{aligned}
& Z_k(\alpha) w^k(\alpha) - \mu^k(\alpha) e \\
& = Z_k(\alpha)(w^k + \alpha \Delta w^k + g^k(\alpha)) - \mu^k(\alpha) e \\
& = Z_k(\alpha)(w^k + \alpha \Delta w^k) + Z_k(\alpha) g^k(\alpha) - \mu^k(\alpha) e \\
& = (Z_k + \alpha \Delta Z_k)(w^k + \alpha \Delta w^k) + Z_k(\alpha) g^k(\alpha) - \mu^k(\alpha) e \\
& = Z_k w^k + \alpha (Z_k \Delta w^k + W_k \Delta z^k) + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha) - \mu^k(\alpha) e \\
& = Z_k w^k + \alpha(-Z_k w^k + \gamma \mu^k e) + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha) - \mu^k(\alpha) e \\
& \qquad \qquad \qquad \qquad \text{(by (44))} \\
& = (1 - \alpha) Z_k w^k + (\alpha \gamma \mu^k - \mu^k(\alpha)) e + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha) \\
& = (1 - \alpha)(Z_k w^k - \mu^k e) + \{(1 - \alpha)\mu^k + \alpha \gamma \mu^k - \mu^k(\alpha)\} e + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha) \\
& = (1 - \alpha)(Z_k w^k - \mu^k e) + \{((1 - \alpha(1 - \gamma))\mu^k - \mu^k(\alpha)\} e + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha) \\
& = (1 - \alpha)(Z_k w^k - \mu^k e) + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha), \\
& \qquad \qquad \qquad \qquad \text{(by } 1 - \gamma = \eta \text{ and (ii) of the lemma)}
\end{aligned}
$$

we have

$$
\begin{aligned}
\|Z_k(\alpha) w^k(\alpha) - \mu^k(\alpha) e\| & = \|(1 - \alpha)(Z_k w^k - \mu^k e) + \alpha^2 \Delta Z_k \Delta w^k + Z_k(\alpha) g^k(\alpha)\| \\
& \le (1 - \alpha) \|Z_k w^k - \mu^k e\| + \alpha^2 \|\Delta Z_k \Delta w^k\| + \|Z_k(\alpha) g^k(\alpha)\| \\
& \le (1 - \alpha) \beta \mu^k + \alpha^2 \|\Delta Z_k \Delta w^k\| + \|Z_k(\alpha) g^k(\alpha)\| \\
& \qquad \qquad \qquad \text{(by (55))} \\
& = \beta \frac{1 - \alpha}{1 - \alpha(1 - \gamma)} \mu^k(\alpha) + \alpha^2 \|\Delta Z_k \Delta w^k\| + \|Z_k(\alpha) g^k(\alpha)\|. \\
& \qquad \qquad \qquad \text{(by (ii) of the lemma and } \eta = 1 - \gamma)
\end{aligned}
$$

The assertion follows from the inequality

$$\frac{1-\alpha}{1-\alpha(1-\gamma)} = 1-\alpha\gamma - \frac{\alpha^2\gamma(1-\gamma)}{1-\alpha(1-\gamma)} \le 1-\alpha\gamma$$

and from the fact that $\mu^k(\alpha) = (1-\alpha\eta)\mu^k > 0$ for every $\alpha \in (0,1)$.
(iv): Since ψ is continuously differentiable on $\Re^{2n}_{++}$, the definition (54) of g implies that

$$\lim_{\alpha\to 0} \frac{\|g^k(\alpha)\|}{\alpha} = 0.$$

Thus, by (50) and (i) above, we have

$$\begin{aligned}
0 \le \lim_{\alpha\to 0} \frac{1}{\alpha}\left\{\alpha^2 \frac{\|\Delta Z_k \Delta w^k\|}{\beta\mu^k(\alpha)} + \frac{\|Z_k(\alpha) g^k(\alpha)\|}{\beta\mu^k(\alpha)}\right\} &\le \lim_{\alpha\to 0}\left\{\alpha \frac{\|\Delta Z_k \Delta w^k\|}{\beta\mu^k(\alpha)} + \frac{\|Z_k(\alpha)\|}{\beta\mu^k(\alpha)} \frac{\|g^k(\alpha)\|}{\alpha}\right\} \\
&= 0 \cdot \frac{\|\Delta Z_k \Delta w^k\|}{\beta\mu^k} + \frac{\|Z_k\|}{\beta\mu^k}\cdot 0 \\
&= 0.
\end{aligned}$$

∎

To determine the step-size $\alpha \in (0,1)$, we use an inexact line search. Let $\rho \in (0,1)$ be a constant and let p_k be the smallest nonnegative integer satisfying

$$(z^k(\rho^{p_k}), w^k(\rho^{p_k})) > 0 \quad \text{and} \quad \|Z_k(\rho^{p_k}) w^k(\rho^{p_k}) - \mu^k(\rho^{p_k}) e\| < \beta\mu(\rho^{p_k}). \tag{57}$$

The existence of such a p_k is guaranteed by (iii) and (iv) of Lemma 5.2. We define the new iterate (z^{k+1}, w^{k+1}) by

$$z^{k+1} := z^k(\rho^{p_k}) \quad \text{and} \quad w^{k+1} := w^k(\rho^{p_k}).$$

Lemma 5.3 *Suppose that f is a P_0 function on $\Re^n_{++}$. Let $\eta + \gamma = 1$.*

(i) *The algorithm described above is well defined.*

(ii) *At each $k = 0, 1, \ldots$, $(z^k, w^k) > 0$ satisfies*

$$r^k = \frac{\mu^k}{\mu^0} r^0 > 0 \quad \textit{and} \quad \|Z_k w^k - \mu^k e\| < \beta\mu^k, \tag{58}$$

and $\{\mu^k\} \subset \Re^n_{++}$ is monotonically decreasing.

(iii) *For each $k = 0, 1, \ldots$, define*

$$(a^k, b^k) := \Psi(z^k, w^k) = (Z_k w^k, r^k).$$

Then there exists a bounded open subset $\Omega \subset \Re^{4n}_{++}$ satisfying the inequalities in (11) and (32) with

$$\omega_1 = \frac{1-\beta}{\mu^0 \max_{i\in 2N}\{r^0_i\}} \quad \textit{and} \quad \omega_2 = \frac{2(1+\beta)}{\mu^0 \min_{i\in 2N}\{r^0_i\}}$$

and

$$(a^k, b^k) \in \Omega.$$

for every $k = 0, 1, \ldots$.

Proof: (i): The assertion follows from the nonsingularity of the matrix (46) and from the existence of an integer p_k satisfying (57) at each iteration k.
(ii): By (i) and (iii) of Lemma 5.2 and by the choice of the step-size $\alpha \in (0,1)$, we see that $(z^k, w^k) > 0$ satisfies

$$r^k = \frac{\mu^k}{\mu^{k-1}} r^{k-1} = \cdots = \frac{\mu^k}{\mu^0} r^0 > 0 \ \text{ and } \ \|Z_k w^k - \mu^k e\| < \beta \mu^k$$

for every $k = 0, 1, \ldots$. Similarly, by (ii) of Lemma 5.2 and by the choice of $\alpha \in (0,1)$, we also see that $\{\mu^k\}$ is monotonically decreasing.
(iii): By (ii) above, $(a^k, b^k) = (Z_k w^k, r^k)$ satisfies

$$0 < \frac{\mu^k}{\mu^0} \min_{i \in 2N} \{r_i^0\} \le b_i^k \le \frac{\mu^k}{\mu^0} \max_{i \in 2N} \{r_i^0\}$$

and

$$0 < (1-\beta)\mu^k < a_i^k < (1+\beta)\mu^k.$$

for every $i \in 2N$ and $k = 0, 1, \ldots$. Since $\mu^k < \mu^0$ for every $k = 1, 2, \ldots$, the above inequalities imply that $\{(a^k, b^k)\} \subset \Re^{4n}_{++}$ is bounded and that

$$0 < \frac{1-\beta}{\mu^0 \max_{i \in 2N}\{r_i^0\}} < \frac{a_{i^1}}{b_{i^2}} < \frac{a_{i^1} + a_{i^3}}{b_{i^2}} < \frac{2(1+\beta)}{\mu^0 \min_{i \in 2N}\{r_i^0\}}$$

for every indices $i^1 \in 2N$, $i^2 \in 2N$ and $i^3 \in 2N$. This completes the proof of (iii). ∎

If Assumption 3.7 is satisfied, Theorem 3.8 and the lemma above guarantee the boundedness of the sequence $\{(z^k, w^k)\} \subset \Psi^{-1}(\Omega)$. Using this fact, we show the global convergence of the algorithm as the theorem below.

Theorem 5.4 *Suppose that Assumption 3.7 is satisfied. Let $\{(z^k, w^k) : \ k = 1, 2, \ldots\}$ be the sequence generated by the algorithm.*

(i) *The sequence $\{(z^k, w^k)\}$ is bounded.*

(ii) *The sequence $\{\mu^k\}$ is monotonically decreasing and converges to 0 as $k \to \infty$.*

(iii) *Every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\{(z^k, w^k)\}$ is a solution of (HCP) with $t^* > 0$.*

(iv) *For every limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\{(z^k, w^k)\}$, $(T_*^{-1} x^*, T_*^{-1} s^*)$ is a complementarity solution of the original problem (CP).*

Proof: (i): It follows from Lemma 5.3 and Theorem 3.8.
(ii): Since we have already seen that $\{\mu^k\} \in \Re_{++}$ is monotonically decreasing in (ii) of Lemma 5.2, it suffices to show that $\lim_{k \to \infty} \mu^k = 0$.

Let $\tilde{\mu} \ge 0$ be a number to which the sequence $\{\mu^k\} \subset \Re_{++}$ converges. Suppose that $\tilde{\mu} > 0$. Then, by (i) above, there exists a subsequence $\{(z^k, w^k)\}_K$ such that

$$(z^k, w^k) \underset{K}{\rightarrow} (\tilde{z}, \tilde{w}) \in \Re^{4n}_{+}$$

where $(\tilde{z}, \tilde{w})$ satisfies

$$\frac{\tilde{z}^T \tilde{z}}{2n} = \tilde{\mu} > 0 \quad \text{and} \quad \|\tilde{Z}\tilde{w} - \tilde{\mu}e\| \leq \beta\tilde{\mu}.$$

Thus we have $(\tilde{z}, \tilde{w}) \in \Re^{4n}_{++}$ and an iteration of the algorithm can be executed at $(\tilde{z}, \tilde{w})$. Let $(\Delta\tilde{z}, \Delta\tilde{w})$ be the direction which is given by the solution of (43) and (44) corresponding to $(\tilde{z}, \tilde{w})$. Then by Lemma 5.2, there exists a $\tilde{\alpha} > 0$ such that for every $\alpha \in (0, \tilde{\alpha})$,

$$\tilde{\mu}(\alpha) = (1 - \alpha\eta)\tilde{\mu} \quad \text{and} \quad \|\tilde{Z}(\alpha)\tilde{w}(\alpha) - \tilde{\mu}(\alpha)e\| < \beta\tilde{\mu}(\alpha)$$

where $\tilde{z}(\alpha)$, $\tilde{w}(\alpha)$, $\tilde{r}(\alpha)$ and $\tilde{\mu}(\alpha)$ are associated with the functions $z^k(\alpha)$, $w^k(\alpha)$, $r^k(\alpha)$ and $\mu^k(\alpha)$ at $(\tilde{z}, \tilde{w})$. Since the Jacobian matrix $\nabla\psi(z)$ is nonsingular and continuous at $\tilde{z}$, we can see that for each $\alpha \in (0, \tilde{\alpha})$,

$$z^k(\alpha) \underset{K}{\rightarrow} \tilde{z}(\alpha), \quad w^k(\alpha) \underset{K}{\rightarrow} \tilde{w}(\alpha), \quad \mu^k(\alpha) \underset{K}{\rightarrow} \tilde{\mu}(\alpha).$$

Thus, there exists an integer $\tilde{p}$ which satisfies that

$$\mu^k(\rho^{\tilde{p}}) = (1 - \rho^{\tilde{p}}\eta)\mu^k \quad \text{and} \quad \|Z_k(\rho^{\tilde{p}})w^k(\rho^{\tilde{p}}) - \mu^k(\rho^{\tilde{p}})e\| < \beta\mu^k(\rho^{\tilde{p}})$$

for every sufficiently large k. Since the choice of p_k forces it to satisfy $p_k \leq \tilde{p}$ for every sufficiently large k, we have

$$\mu^{k+1} = \mu^k(\rho^{p_k}) = (1 - \rho^{p_k}\eta)\mu^k \leq (1 - \rho^{\tilde{p}}\eta)\mu^k$$

and μ^k decreases at least by the factor $1 - \rho^{\tilde{p}}\eta$ for every sufficiently large k. This contradicts the assumption that $\mu^k \rightarrow \tilde{\mu} > 0$.
(iii): The proof is analogous to the one of Theorem 3.6 (ii). Let $(\bar{z}^*, \bar{w}^*) = (\bar{x}^*, \bar{t}^*, \bar{s}^*, \bar{u}^*)$ be a complementarity solution of (HCP) with $\bar{t}^* > 0$, whose existence has been ensured by (vi) of Lemma 2.2. Then, for every k, we have

$$\begin{aligned}
&(z_i^k - \bar{z}_i^*)(\psi_i(z^k) - \psi_i(\bar{z}^*)) \\
&= (z_i^k - \bar{z}_i^*)[(w_i^k - r^k) - \bar{w}_i^*] \\
&= (z_i^k - \bar{z}_i^*)[(w_i^k - \bar{w}_i^*) - r^k] \\
&= (z_i^k - \bar{z}_i^*)(w_i^k - \bar{w}_i^*) - (z_i^k - \bar{z}_i^*)r^k \\
&= [z_i^k w_i^k - (z_i^k \bar{w}_i^* + \bar{z}_i^* w_i^k) + \bar{z}_i^* \bar{w}_i^*] - z_i^k r_i^k + \bar{z}_i^* r_i^k \\
&= [z_i^k w_i^k - (z_i^k \bar{w}_i^* + \bar{z}_i^* w_i^k)] - z_i^k r_i^k + \bar{z}_i^* r_i^k \\
&\leq (z_i^k w_i^k + \bar{z}_i^* r_i^k) - (z_i^k \bar{w}_i^* + \bar{z}_i^* w_i^k).
\end{aligned} \tag{59}$$

Here, by (ii) of Lemma 5.3, (z^k, w^k) satisfies (58) and hence

$$0 < z_i^k w_i^k \leq (1 + \beta)\mu^k \quad \text{and} \quad 0 < r_i^k \leq \frac{r^0}{\mu^0}\mu^k.$$

Let us define

$$\delta^k = \mu^k \left(1 + \beta + \max\left\{\bar{z}_i^* \frac{r_i^0}{\mu^0} : \; i \in 2N\right\}\right) > 0. \tag{60}$$

Then by (59),

$$(z_i^k - \bar{z}_i^*)(\psi_i(z^k) - \psi_i(\bar{z}^*)) \le \delta^k - (z_i^k \bar{w}_i^* + \bar{z}_i^* w_i^k) \quad (i \in 2N)$$

and by Theorem 3.8 and (iii) of Lemma 3.2, we have

$$\sum_{i \in 2N} (\bar{z}_i^* w_i^k + z_i^k \bar{w}_i^*) \le (1 + 4\kappa_C) 2n\delta^k.$$

On the other hand, it follows from (58) that $Z_k w^k \ge (1-\beta)\mu^k e > 0$ and hence,

$$w_i^k \ge (1-\beta)\mu^k / z_i^k \quad \text{and} \quad z_i^k \ge (1-\beta)\mu^k / w_i^k \quad (i \in 2N).$$

Thus we obtain that

$$\sum_{i \in 2N} (1-\beta)\mu^k \left(\frac{\bar{z}_i^*}{z_i^k} + \frac{\bar{w}_i^*}{w_i^k} \right) \le (1 + 4\kappa_C) 2n\delta^k.$$

By the definition (60) of $\delta^k > 0$, the above inequality implies that

$$\sum_{i \in 2N} \left(\frac{\bar{z}_i^*}{z_i} + \frac{\bar{w}_i^*}{w_i} \right) \le \frac{1}{1-\beta}(1 + 4\kappa_C) 2n \left(1 + \beta + \max\left\{ \bar{z}_i^* \frac{r_i^0}{\mu^0} : \ i \in 2N \right\} \right).$$

Define

$$\delta^* = \frac{1}{1-\beta}(1 + 4\kappa_C) 2n \left(1 + \beta + \max\left\{ \bar{z}_i^* \frac{r_i^0}{\mu^0} : \ i \in 2N \right\} \right).$$

Then we see that

$$\frac{\bar{z}_i^*}{z_i^k} \le \delta^* \quad \text{and} \quad \frac{\bar{w}_i^*}{w_i^k} \le \delta^* \quad (i \in N)$$

and

$$\frac{\bar{x}_i^*}{x_i^k} \le \delta^*, \quad \frac{\bar{t}_i^*}{t_i^k} \le \delta^*, \quad \frac{\bar{s}_i^*}{s_i^k} \le \delta^*, \quad \frac{\bar{u}_i^*}{u_i^k} \le \delta^* \quad (i \in N)$$

for every $k = 1, 2, \ldots$. Since $\bar{t}^* > 0$, the above inequalities ensure that any limit point $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of $\{(z^k, w^k)\}$ satisfies $t^* > 0$.
(iv): It follows from (iii) above and (iii) of Lemma 2.2. ∎

6 Convergence rate of the algorithm

In this section, we discuss the convergence rate of the algorithm described in Section 5. As we have shown in Lemma 5.2, the rate of convergence depends on the size of $\alpha \in (0, 1)$ satisfying the inequalities

$$(z^k(\alpha), w^k(\alpha)) > 0 \quad \text{and} \quad \|Z_k(\alpha) w^k(\alpha) - \mu^k(\alpha) e\| < \beta\mu^k(\alpha) \tag{61}$$

at each k. To derive a lower bound of such step sizes, we impose an assumption on the smoothness of the function ψ.

Assumption 6.1 *There exists a $\lambda > 0$ for which*

$$\|Z[\psi(z+\alpha\Delta z) - \psi(z) - \alpha\nabla\psi(z)\Delta z]\| \leq \lambda\alpha^2\|\Delta Z\nabla\psi(z)\Delta z\| \tag{62}$$

whenever $\Delta z \in \Re^{2n}$, $z > 0$, $\alpha \in (0,1)$ and $\alpha\|Z^{-1}\Delta z\| \leq 1$.

We discuss the assumption above in terms of the original problem (CP) in the last of this section.

Suppose that Assumption 6.1 holds. Then, for every $\alpha \in (0,1)$ satisfying $\alpha\|Z^{-1}\Delta z\| \leq 1$, the term $\|Z_k(\alpha)g^k(\alpha)\|$ appeared in (iii) of Lemma 5.2 is bounded by

$$\begin{aligned}
\|Z_k(\alpha)g^k(\alpha)\| &= \|Z_k(\alpha)[\psi(z^k+\alpha\Delta z) - \psi(z^k) - \alpha\nabla\psi(z^k)\Delta z^k]\| \\
&\qquad\qquad\qquad\qquad \text{(by (54))} \\
&\leq \|Z_k^{-1}Z_k(\alpha)\|\|Z_k[\psi(z^k+\alpha\Delta z) - \psi(z^k) - \alpha\nabla\psi(z^k)\Delta z^k]\| \\
&= \|I + \alpha Z^{-1}\Delta Z\|\|Z_k[\psi(z^k+\alpha\Delta z) - \psi(z^k) - \alpha\nabla\psi(z^k)\Delta z^k]\| \\
&\leq 2\|Z_k[\psi(z^k+\alpha\Delta z) - \psi(z^k) - \alpha\nabla\psi(z^k)\Delta z^k]\| \\
&\qquad\qquad\qquad\qquad \text{(by } \alpha\|Z^{-1}\Delta z\| \leq 1) \\
&\leq 2\lambda\alpha^2\|\Delta Z_k\nabla\psi(z^k)\Delta z^k\| \\
&= 2\lambda\alpha^2\|\Delta Z_k\Delta w^k\|. \\
&\qquad\qquad\qquad\qquad \text{(by (43))}
\end{aligned}$$

The inequality above gives a new bound for the term $\|Z_k(\alpha)w^k(\alpha) - \mu^k(\alpha)e\|$ instead of the one in (iii) of Lemma 5.2:

$$\|Z_k(\alpha)w^k(\alpha) - \mu^k(\alpha)e\| \leq \beta\mu^k(\alpha)\left\{(1-\alpha\gamma) + \alpha^2(1+2\lambda)\frac{\|\Delta Z_k\Delta w^k\|}{\beta(1-\alpha\eta)\mu^k}\right\}. \tag{63}$$

Our intention is now to estimate the value of the term $\|\Delta Z_k\Delta w^k\|$. The lemma below is crucial in our analysis.

Lemma 6.2 *Suppose that Assumption 3.7 holds. Then there exists a $\kappa_C \geq 0$ for which the Jacobian matrix $\nabla\psi(z)$ of the function ψ is a $P_*(\kappa_C)$ matrix at every z in the sequence $\{z^k : k = 1, 2, \ldots\}$ generated by the algorithm.*

Proof: By (iii) of Lemma 5.3, there exists a bounded open set Ω satisfying (11), (32) and that

$$\Psi(\{(z^k, w^k)\}) \subset \Omega.$$

Thus, by Theorem 3.8, there exists a set $C = C(\pi) \subset \Re^{4n}_{++}$ for some $\pi \in (0,1)$ which satisfies

$$\{(z^k, w^k)\} \subset \Psi^{-1}(\Omega) \subset \Re^n_{++} \times C \times \Re^{2n}_{++}.$$

By (ii) of Lemma 2.1, this implies that the function ψ is a $\mathrm{P}_*(\kappa_C)$ function on the set $\Re^n_{++} \times C \times \Re^{2n}_{++}$. Note that the set $C(\pi)$ is open and convex by its definition (3). Therefore, by Corollary 4.7, the Jacobian matrix $\nabla\psi(z)$ is a $\mathrm{P}_*(\kappa_C)$ matrix at every $z \in \{z^k\} \subset \Re^n_{++} \times C$. ∎

Using the above lemma, we show the following results.

Lemma 6.3 *Suppose that Assumption 3.7 holds. Let us choose $\beta \in (0,1)$, $\eta \in (0,1)$ and $\gamma \in (0,1)$ so that*

$$\eta + \gamma = 1 \quad \text{and} \quad \beta \le \sqrt{2n}\eta.$$

Then the direction $(\Delta z^k, \Delta w^k)$ given by the solution of (43) and (44) satisfies the following inequalities.

(i)

$$\|D_k \Delta w\|^2 + \|D_k^{-1}\Delta z^k\|^2 \le \frac{\mu^k}{1-\beta}\left\{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2\eta^2\right\}$$

where $D_k = Z_k^{1/2} W_k^{-1/2}$.

(ii)

$$\|Z_k^{-1}\Delta z^k\| \le \frac{1}{1-\beta}\sqrt{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2\eta^2}.$$

(iii)

$$\|\Delta Z_k \Delta w^k\| \le \frac{\mu^k}{2(1-\beta)}\left\{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2\eta^2\right\}.$$

Proof: (i): Define

$$\Delta \bar{w}^k = \Delta w^k + \eta r^k.$$

The equations (43) and (44) can be represented using $\Delta \bar{w}^k$ as

$$\Delta \bar{w}^k - \nabla\psi(z^k)\Delta z^k = 0, \tag{64}$$
$$Z_k \Delta \bar{w}^k + W_k \Delta z^k = \gamma\mu^k e - Z_k w^k + \eta Z_k r^k. \tag{65}$$

Let

$$h^k := (Z_k W_k)^{-1/2}(\gamma\mu^k e - Z_k w^k + \eta Z_k r^k). \tag{66}$$

Then the equations (64) and (65) are equivalent to

$$\begin{aligned} \Delta \bar{w}^k - \nabla\psi(z^k)\Delta z^k &= 0, \\ D_k \Delta \bar{w}^k + D_k^{-1}\Delta z^k &= h^k. \end{aligned}$$

Since $\nabla\psi(z^k)$ is a $\mathrm{P}_*(\kappa_C)$ matrix by Lemma 6.2, as a property of the $\mathrm{P}_*(\kappa_C)$ property (see, e.g., Lemma 3.4 of [9]), we have

$$(\Delta z^k)^T \Delta \bar{w}^k \ge -\kappa_C \|h^k\|^2. \tag{67}$$

Note that Lemma 5.1 with $\eta + \gamma = 1$ and (64) imply that

$$\begin{aligned} (\Delta z^k)^T \Delta w^k &= (\Delta z^k)^T \nabla\psi(z^k)\Delta z^k \\ &= (\Delta z^k)^T \Delta \bar{w}^k. \end{aligned}$$

Thus, by (44) and (67), we see that

$$\begin{aligned}
\|D_k\Delta w^k\|^2 + \|D_k^{-1}\Delta z^k\|^2 &= \|D_k\Delta w^k + D_k^{-1}\Delta z^k\|^2 - 2(\Delta z^k)^T\Delta w^k \\
&= \|(Z_kW_k)^{-1/2}(Z_k\Delta w^k + W_k\Delta z^k)\|^2 - 2(\Delta z^k)^T\Delta w^k \\
&= \|(Z_kW_k)^{-1/2}(\gamma\mu^k e - Z_kw^k)\|^2 - 2(\Delta z^k)^T\Delta w^k \\
&\le \|(Z_kW_k)^{-1/2}(\gamma\mu^k e - Z_kw^k)\|^2 + 2\kappa_C\|h^k\|^2. \qquad (68)
\end{aligned}$$

The following inequalities can be obtained from the facts $\|Z_kw^k - \mu^k e\| \le \beta\mu^k$ and $r^k > 0$:

$$\begin{aligned}
\|\gamma\mu^k e - Z_kw^k\|^2 &= \|(Z_kw^k - \mu^k e) + (1-\gamma)\mu^k e\|^2 \\
&= \|(Z_kw^k - \mu^k e) + \eta\mu^k e\|^2 \\
&\qquad\qquad (\text{by } \eta + \gamma = 1) \\
&= \|Z_kw^k - \mu^k e\|^2 + \eta\mu^k e^T(Z_kw^k - \mu^k e) + (\eta\mu^k)^2\ \|e\|^2 \\
&= \|Z_kw^k - \mu^k e\|^2 + (\eta\mu^k)^2 2n \\
&\qquad\qquad (\text{by (42) }) \\
&\le (\beta\mu^k)^2 + (\eta\mu^k)^2 2n \\
&\qquad\qquad (\text{by } \|Z_kw^k - \mu^k e\| \le \beta\mu^k\ ) \\
&= (\beta^2 + (2n)\eta^2)(\mu^k)^2,
\end{aligned}$$

$$\begin{aligned}
\|Z_kr^k\| &\le \|Z_kr^k\|_1 \\
&= e^T(Z_kr^k) \\
&\qquad\qquad (\text{by } z^k > 0 \text{ and } r^k > 0) \\
&= (z^k)^T(w^k - \psi(z^k)) \\
&\qquad\qquad (\text{by (42) }) \\
&= (z^k)^T w^k \\
&\qquad\qquad (\text{by (i) of Lemma 2.2 }) \\
&= (2n)\mu^k \\
&\qquad\qquad (\text{by (42) })
\end{aligned}$$

and

$$(1-\beta)\mu^k \le \|Z_kW_k\| \le (1+\beta)\mu^k. \qquad (\text{by } \|Z_kw^k - \mu^k e\| \le \beta\mu^k\ ) \qquad (69)$$

Combining the above inequalities with (68) and (66), we have

$$\begin{aligned}
&\|D_k\Delta w\|^2 + \|D_k^{-1}\Delta z^k\|^2 \\
&\le \|(Z_kW_k)^{-1/2}(\gamma\mu^k e - Z_kw^k)\|^2 + 2\kappa_C\|h^k\|^2 \\
&= \|(Z_kW_k)^{-1/2}(\gamma\mu^k e - Z_kw^k)\|^2 + 2\kappa_C\|(Z_kW_k)^{-1/2}(\gamma\mu^k e - Z_kw^k + \eta Z_kr^k)\|^2 \\
&\le \|Z_kW_k\|^{-1}\|\gamma\mu^k e - Z_kw^k\|^2 + 2\kappa_C\|Z_kW_k\|^{-1}(\|\gamma\mu^k e - Z_kw^k\| + \eta\|Z_kr^k\|)^2 \\
&= \|Z_kW_k\|^{-1}\left\{\|\gamma\mu^k e - Z_kw^k\|^2 + 2\kappa_C(\|\gamma\mu^k e - Z_kw^k\| + \eta\|Z_kr^k\|)^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(1-\beta)\mu^k}\left\{(\beta^2+(2n)\eta^2)(\mu^k)^2+2\kappa_C\left(\sqrt{\beta^2+(2n)\eta^2}\mu^k+\eta(2n)\mu^k\right)^2\right\}\\
&= \frac{\mu^k}{1-\beta}\left\{(\beta^2+(2n)\eta^2)+2\kappa_C\left(\sqrt{\beta^2+(2n)\eta^2}+(2n)\eta\right)^2\right\}\\
&\leq \frac{\mu^k}{1-\beta}\left\{(\beta^2+(2n)\eta^2)+2\kappa_C(4n)^2\eta^2\right\}
\end{aligned}
$$

where the last inequality follows from the fact that the assumption $\beta \leq \sqrt{2n}\eta$ implies

$$
\begin{aligned}
\left(\sqrt{\beta^2+(2n)\eta^2}+(2n)\eta\right)^2 &\leq \left(\sqrt{(2n)\eta^2+(2n)\eta^2}+(2n)\eta\right)^2\\
&= \left(2\sqrt{n}\eta+(2n)\eta\right)^2\\
&\leq (4n\eta)^2.
\end{aligned}
$$

(ii): Since

$$
\begin{aligned}
\|Z_k^{-1}\Delta z^k\| &= \|(Z_kW_k)^{-1/2}D_k^{-1}\Delta z^k\|\\
&\leq \|(Z_kW_k)\|^{-1/2}\|D_k^{-1}\Delta z^k\|\\
&\leq \|(Z_kW_k)\|^{-1/2}\sqrt{\|D_k\Delta w^k\|^2+\|D_k^{-1}\Delta z^k\|^2},
\end{aligned}
$$

the assertion follows from (69) and (i) above.

(iii): The inequality is obtained from (i) above and the fact that

$$
\begin{aligned}
\|\Delta Z_k\Delta w^k\| &= \|D_k^{-1}\Delta Z_kD_k\Delta w^k\|\\
&\leq \|D_k^{-1}\Delta z^k\|\|D_k\Delta w^k\|\\
&\leq (\|D_k\Delta w^k\|^2+\|D_k^{-1}\Delta z^k\|^2)/2.
\end{aligned}
$$

∎

Substituting (iii) of Lemma 6.3 into the inequality (63), we obtain

$$
\|Z_k(\alpha)w^k(\alpha)-\mu^k(\alpha)e\| \leq \beta\mu^k(\alpha)\left\{(1-\alpha\gamma)+\alpha^2(1+2\lambda)\frac{(\beta^2+(2n)\eta^2)+8\kappa_C(2n)^2\eta^2}{2(1-\beta)\beta(1-\alpha\eta)}\right\}.
$$

Since $1-\eta < 1-\alpha\eta$ for every $\alpha \in (0,1)$, the above inequality implies that if

$$
\tilde{\alpha} < \frac{\gamma 2(1-\beta)\beta(1-\eta)}{(1+2\lambda)\{(\beta^2+(2n)\eta^2)+8\kappa_C(2n)^2\eta^2\}}
$$

then

$$
\|Z_k(\alpha)w^k(\alpha)-\mu^k(\alpha)e\| < \beta\mu^k(\alpha) \tag{70}
$$

for every $\alpha \in (0,\tilde{\alpha}]$. On the other hand, by (ii) of Lemma 6.3, if

$$
\tilde{\alpha} < \frac{1-\beta}{\sqrt{(\beta^2+(2n)\eta^2)+8\kappa_C(2n)^2\eta^2}}
$$

then

$$\alpha \| Z_k^{-1} \Delta z^k \| < 1$$

and hence

$$z^k + \alpha \Delta z^k = Z_k (e + \alpha Z_k^{-1} \Delta z^k) > 0 \tag{71}$$

for every $\alpha \in (0, \tilde{\alpha}]$. Combining the above results, we can conclude that if

$$\tilde{\alpha} = \min \left\{ \frac{\gamma(1-\beta)\beta(1-\eta)}{(1+2\lambda)\{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2 \eta^2\}}, \; \frac{1-\beta}{2\sqrt{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2 \eta^2}} \right\} \tag{72}$$

then (70) and (71) hold, and hence

$$w^k(\alpha) > 0$$

for every $\alpha \in (0, \tilde{\alpha}]$. Thus we obtain the following theorem.

Theorem 6.4 *Suppose that Assumptions 3.7 and 6.1 hold. Let us choose $\beta \in (0,1)$, $\eta \in (0,1)$ and $\gamma \in (0,1)$ so that*

$$\eta + \gamma = 1 \quad \text{and} \quad \beta \le \sqrt{2n}\eta.$$

Then there exists a $\tilde{\alpha} > 0$ for which (61) holds for every $\alpha \in (0, \tilde{\alpha}]$ and at every k. An example of such a $\tilde{\alpha}$ is given by (72). In addition, the values μ^k and $\|r^k\|$ are reduced by the factor $1 - \tilde{\alpha}\eta$ at each k.

Now we find more specific values of $\beta \in (0,1)$, $\eta \in (0,1)$, $\gamma \in (0,1)$ and $\alpha \in (0,1)$ in order to derive a complexity bound of the algorithm. Let us choose

$$\eta = \frac{1}{\sqrt{2n}}, \quad \gamma = 1 - \eta, \quad \beta = \frac{1}{2}.$$

Then β, η and γ satisfy the assumptions $\eta + \gamma = 1$ and $\beta \le \sqrt{2n}\eta$ in Lemma 6.3 and we obtain a lower bound of the right hand side of (72) as follows

$$\begin{aligned}
&\min \left\{ \frac{\gamma 2(1-\beta)\beta(1-\eta)}{(1+2\lambda)\{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2 \eta^2\}}, \; \frac{1-\beta}{2\sqrt{(\beta^2 + (2n)\eta^2) + 8\kappa_C (2n)^2 \eta^2}} \right\} \\
&= \min \left\{ \frac{(1/2)(1-\eta)^2}{(1+2\lambda)\{(5/4) + 16n\kappa_C\}}, \; \frac{1/2}{2\sqrt{(5/4) + 8\kappa_C (2n)}} \right\} \\
&\ge \frac{(1-\eta)^2}{4(1+2\lambda)\{(5/4) + 16n\kappa_C\}} \\
&\qquad\qquad (\text{since } (5/4) + 16n\kappa_C > 1) \\
&\ge \frac{1}{36(1+2\lambda)\{(5/4) + 16n\kappa_C\}} \\
&\qquad\qquad (\text{since } \eta = 1/\sqrt{2n} < 2/3).
\end{aligned}$$

By Theorem 6.4, we directly obtain a complexity bound of the algorithm as follows:

Theorem 6.5 *Suppose that Assumptions 3.7 and 6.1 hold. Then the number K of iteration required until $\mu^K \le \epsilon$ is*

$$K = \mathcal{O}\left(\sqrt{n}\max\{1,\lambda\}\max\{1,n\kappa_C\}\log\left(\frac{\mu^0}{\epsilon}\right)\right).$$

Unfortunately, Assumption 6.1 does not necessarily hold even if the original function f satisfies a Lipschitz-type condition, even if f is linear. To see this, let us define

$$y := T^{-1}x, \quad y(\alpha) := (T+\alpha\Delta T)^{-1}(x+\alpha\Delta x), \quad \Delta y(\alpha) := y(\alpha) - y. \tag{73}$$

Here $\Delta y(\alpha)$ can be calculated as

$$\begin{aligned}
\Delta y(\alpha) &= (T+\alpha\Delta T)^{-1}(x+\alpha\Delta x) - T^{-1}x \\
&= (T+\alpha\Delta T)^{-1}\left\{(x+\alpha\Delta x) - (T+\alpha\Delta T)T^{-1}x\right\} \\
&= (T+\alpha\Delta t)^{-1}(\alpha\Delta x - \alpha(\Delta T)T^{-1}x) \\
&= \alpha(T+\alpha\Delta t)^{-1}(\Delta x - (\Delta T)T^{-1}x) \\
&= \alpha(T+\alpha\Delta t)^{-1}(\Delta x - T^{-1}X\Delta t).
\end{aligned} \tag{74}$$

By (40), we have

$$\begin{aligned}
&\alpha\nabla\psi(z)\Delta z \\
&= \alpha\begin{pmatrix} T\nabla f(y)T^{-1}\Delta x + F(y)\Delta t - T\nabla f(y)T^{-2}X\Delta t \\ -F(y)\Delta x - X\nabla f(y)T^{-1}\Delta x + X\nabla f(y)T^{-2}X\Delta t \end{pmatrix} \\
&= \alpha\begin{pmatrix} F(y)\Delta t + T\nabla f(y)T^{-1}(\Delta x - T^{-1}X\Delta t) \\ -F(y)\Delta x - X\nabla f(y)T^{-1}(\Delta x - T^{-1}X\Delta t) \end{pmatrix} \\
&= \begin{pmatrix} \alpha F(y)\Delta t + T\nabla f(y)T^{-1}\alpha(\Delta x - T^{-1}X\Delta t) \\ -\alpha F(y)\Delta x - X\nabla f(y)T^{-1}\alpha(\Delta x - T^{-1}X\Delta t) \end{pmatrix} \\
&= \begin{pmatrix} \alpha F(y)\Delta t + T\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha) \\ -\alpha F(y)\Delta x - X\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha) \end{pmatrix}
\end{aligned}$$

where the last equality follows from (74). Substituting the above and (1) into (62), we see that

$$\begin{aligned}
&Z\left[\psi(z+\alpha\Delta z) - \psi(z) - \alpha\nabla\psi(z)\Delta z\right] \\
&= \begin{pmatrix} X\left[(T+\alpha\Delta T)f(y(\alpha)) - Tf(y) - \alpha F(y)\Delta t - T\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ T\left[-(X+\alpha\Delta X)f(y(\alpha)) + Xf(y) + \alpha F(y)\Delta x + X\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \end{pmatrix} \\
&= \begin{pmatrix} X\left[(T+\alpha\Delta T)f(y(\alpha)) - (T+\alpha\Delta T)f(y) - T\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ T\left[-(X+\alpha\Delta X)f(y(\alpha)) + (X+\alpha\Delta X)f(y) + X\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} X\left[(T+\alpha\Delta T)(f(y(\alpha))-f(y))-T\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ T\left[-(X+\alpha\Delta X)(f(y(\alpha))-f(y))+X\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \end{pmatrix}.$$

Using the facts

$$\begin{aligned} X(T+\alpha\Delta T) &= T(T^{-1}X)T(I+\alpha T^{-1}\Delta T) = T^2(I+\alpha T^{-1}\Delta T)Y, \\ T(X+\alpha\Delta X) &= T^2(T^{-1}X)(I+\alpha X^{-1}\Delta X) = T^2(I+\alpha X^{-1}\Delta X)Y, \end{aligned}$$

induced by (73), the upper term can be represented as

$$\begin{aligned} & X\left[(T+\alpha\Delta T)(f(y(\alpha))-f(y))-T\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ &= X\left[(T+\alpha\Delta T)(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right. \\ &\qquad \left.+(T+\alpha\Delta T)\nabla f(y)\Delta y(\alpha)-T\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ &= X\left[(T+\alpha\Delta T)(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right. \\ &\qquad \left.+\alpha\Delta T\nabla f(y)\Delta y(\alpha)-T\nabla f(y)T^{-1}(\alpha\Delta T)\Delta y(\alpha)\right] \\ &= T^2\left[(I+\alpha T^{-1}\Delta T)Y(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right] \\ &\qquad +\alpha XT\left[T^{-1}\Delta T\nabla f(y)\Delta y(\alpha)-\nabla f(y)T^{-1}\Delta T\Delta y(\alpha)\right] \\ &= T^2\left[(I+\alpha T^{-1}\Delta T)Y(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right] \\ &\qquad +\alpha XT\left[T^{-1}\Delta T\nabla f(y)-\nabla f(y)T^{-1}\Delta T\right]\Delta y(\alpha) \end{aligned}$$

and the lower term turns out to be

$$\begin{aligned} & T\left[-(X+\alpha\Delta X)(f(y(\alpha))-f(y))-X\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ &= T\left[-(X+\alpha\Delta X)(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right. \\ &\qquad \left.-(X+\alpha\Delta X)\nabla f(y)\Delta y(\alpha)-X\nabla f(y)T^{-1}(T+\alpha\Delta T)\Delta y(\alpha)\right] \\ &= T\left[-(X+\alpha\Delta X)(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right. \\ &\qquad \left.-\alpha\Delta X\nabla f(y)\Delta y(\alpha)-X\nabla f(y)T^{-1}(\alpha\Delta T)\Delta y(\alpha)\right] \\ &= T^2\left[-(I+\alpha X^{-1}\Delta X)Y(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right] \\ &\qquad -\alpha T\left[\Delta X\nabla f(y)\Delta y(\alpha)+X\nabla f(y)T^{-1}(\alpha\Delta T)\Delta y(\alpha)\right] \\ &= T^2\left[-(I+\alpha X^{-1}\Delta X)Y(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right] \\ &\qquad -\alpha T\left[(\Delta X-T^{-1}\Delta T)\nabla f(y)\Delta y(\alpha)\right]-\alpha T\left[T^{-1}\Delta T\nabla f(y)\Delta y(\alpha)-\nabla f(y)T^{-1}\Delta T\Delta y(\alpha)\right] \\ &= T^2\left[-(I+\alpha X^{-1}\Delta X)Y(f(y(\alpha))-f(y)-\nabla f(y)\Delta y(\alpha))\right] \\ &\qquad -\alpha T\left[(T+\alpha\Delta T)\Delta Y(\alpha)\nabla f(y)\Delta y(\alpha)\right]-\alpha T\left[T^{-1}\Delta T\nabla f(y)-\nabla f(y)T^{-1}\Delta T\right]\Delta y(\alpha). \end{aligned}$$

The above equations show that if the term

$$\left\|T^{-1}\Delta T\nabla f(y)-\nabla f(y)T^{-1}\Delta T\right\| \tag{75}$$

is sufficiently small to be ignored, then Assumption 6.1 holds under a suitable smoothness condition on f, e.g., the function f satisfies

$$\|Y\left[f(y(\alpha)) - f(y) - \nabla f(y)\Delta y(\alpha)\right]\| \leq \lambda_f \alpha^2 \|\Delta Y(\alpha)\nabla f(y)\Delta y(\alpha)\|$$

for some $\lambda > 0$. As we have shown in Theorem 5.4, the generated sequence $\{(z^k, w^k) = (x^k, t^k, s^k, u^k)\}$ converges to a solution $(z^*, w^*) = (x^*, t^*, s^*, u^*)$ of (HCP) with $t^* > 0$. The fact implies that

$$\lim_{k\to\infty} \|(T^k)^{-1}\Delta T^k\| = 0$$

and the term (75) vanishes near the solution by the continuity of $\nabla f(y)$ on $\Re^n_+$. Thus, in general, we may consider that Theorem 6.5 gives a locally convergent rate of the algorithm.

7 Concluding remarks

In this paper, we have provided a new homogeneous model which can be applied to P_0 and P_* nonlinear complementarity problems. We have discussed the existence of the trajectory and its limiting behavior under suitable assumptions. An associated algorithm has been also proposed for solving the strictly feasible P_* complementarity problem. We have shown its global convergence property and derived its convergence rate assuming a smoothness condition on the homogeneous function used in the model.

Comparing to the monotone case proposed in [1], there is a lack of discussion concerning how we certify the infeasibility of the problem. In [1], the authors define the residual function associated with their model as follows:

$$\begin{aligned}
\bar{r}_s(x, \tau, s, \kappa) &:= s - \tau f(x/\tau), \\
\bar{r}_u(x, \tau, s, \kappa) &:= \kappa + x^T f(x/\tau), \\
\bar{r}(x, \tau, s, \kappa) &:= (\bar{r}_s(x, \tau, s, \kappa), \bar{r}_u(x, \tau, s, \kappa))
\end{aligned}$$

For the monotone function f, the image $\bar{r}(\Re^{4n}_{++})$ of the function $\bar{r}$ is convex, which is just a key ingredient to derive a certification of the infeasibility. Meanwhile, throughout our analysis, we have only used the fact that the corresponding set $r(\Re^{4n}_{++})$ contains the positive orthant $\Re^{2n}_{++}$, which holds regardless of the property of f (see (iv) of Lemma 2.3). Thus, it might be an issue which merits further research to examine the property of the set $r(\Re^{4n}_{++})$ for the P_0 or the P_* function f.

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